Higher-loop Resummation in QCD (F)APT

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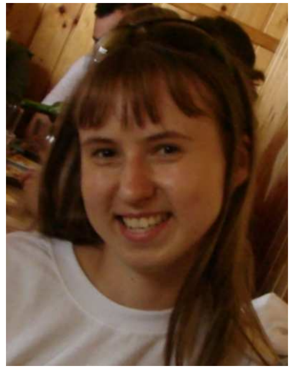
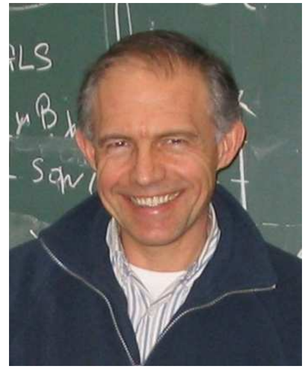
OUTLINE

- Intro: Asymptotic Series in Perturbative QFT
- APT and FAPT
- Resummation in APT and FAPT
  - One-loop case
  - Two-loop case
  - Three-loop case
- Applications: Resummation for Adler function $D(Q^2)$
- Applications: Higgs decay $H^0 \to b\bar{b}$
- Conclusions

Collaborators & Publications

Collaborators:





S. V. Mikhailov	D. V. Shirkov	I. V. Potapova	N. G. Stefanis
JINR (Dubna)	JINR (Dubna)	JINR (Dubna)	RUB (Bochum)

Publications:

- A. B.&Mikhailov — Solovtsov Memorial Seminar, Dubna, Jan. 17–18, 2008, Dubna: JINR (2008) pp. 119–133
- A. B. — Phys. Part. Nucl. 40 (2009) 715
- A. B., Mikhailov, Stefanis — JHEP 1006 (2010) 085
- A. B.&Shirkov — ArXiv:1102.2380[hep-ph]
- A. B.&Potapova — ArXiv:1108.6300[hep-ph]

Asymptotic Series

in

Perturbative QFT

Strength and Weakness of Pert. QFT

A lot of successive pert. calculations in QM and QFT.
Practically, it is synonym of Quantum Theory.
Feynman diagrams became a symbol of QFT.

Nevertheless, power expansion of the quantum amplitude $C(\alpha)$ is not convergent.

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Due to

- **Essential singularity at $\alpha = 0$**
- **Factorial growth of coefficients $c_k \sim k!$**

Singularity at $g = 0$*, factorial growth* $c_k \sim k!$

For illustration, take the 0-dim analog $I(g) = \int\limits_{-\infty}^{\infty} e^{-x^2 - g x^4}\, dx$

Expanding it in power-in-g series:

$I(g) \sim \sum\limits_{k=0} (-g)^k I_k$ with $I_k = \dfrac{\Gamma(2k + 1/2)}{\Gamma(k+1)} \to 2^k\, k!$

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Meanwhile, $I(g)$ can be expressed via MacDonald function

$$I(g) = \frac{1}{\sqrt{2g}}\, e^{1/8g}\, K_{1/4}\left(\frac{1}{8g}\right)$$

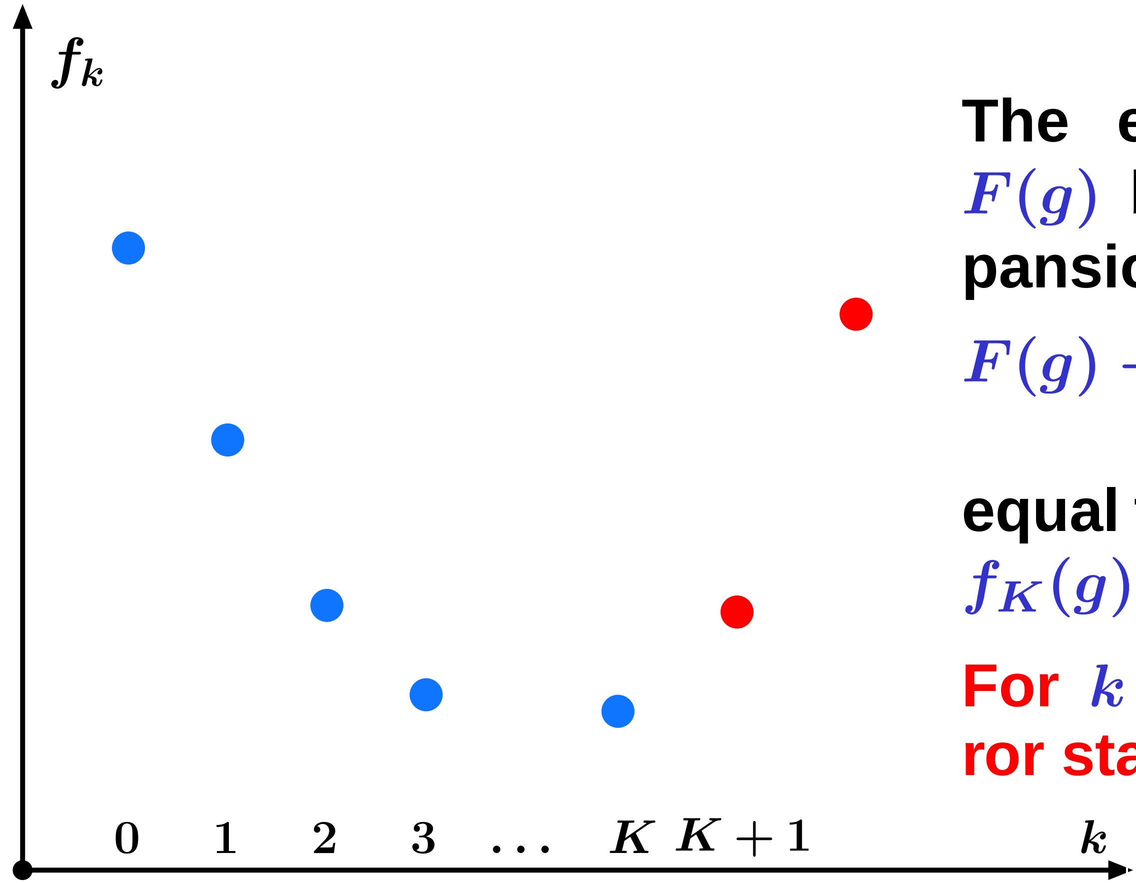
with known analytic properties in complex g plane.

It has an essential singularity $e^{-1/8g}$ near the origin:

$$I(g) = \sqrt{\pi} - \frac{g}{\sqrt{2\pi}} \int_0^{\infty} \frac{d\gamma\, \exp(-1/4\gamma)}{\gamma(g+\gamma)}$$

Asymptotic Series and 'Practic. Convergence'

The Henry Poincaré (end of XIX) analysis of Asymptotic Series (AS) can be summed as follows:
AS can be used for obtaining **quantitative information** on expanded function.



The error of approximating $F(g)$ by first K terms of expansion, $F_K(g)$,

$$F(g) \to F_K(g) = \sum_{k \leq K} f_k(g)$$ is

equal to the last detained term $f_K(g)$.

For $k \geq K+1$ truncation error starts to grow!

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For the power AS, $f_k(g) = f_k\, g^k$ with factorial growth $f_k \sim k!$
absolute values of $f_k(g)$ cease to diminish at $k \sim 1/g$.
This yields to the natural best possible accuracy of a given AS
(in contrast to convergent series!)

Asymptotic Series and ‘Practic. Convergence’

$$I(g) = \int_{-\infty}^{\infty} e^{-x^2 - g x^4}\, dx \quad ? = ? \quad \sum_{k\geq 0} I_k\,(-g)^k$$

g	K	$(-g)^K\, I_K$	$(-g)^{K+1}\, I_{K+1}$	$\Delta_K I(g)$
0.07	7	$-0.04(2\%)$	$+0.07(4.4\%)$	1.4%
0.07	9	$-0.17(10\%)$	$+0.42(25\%)$	7%

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Thus, $K_*(g=0.07)=7$ **and** $K_*(g=0.15)=2$**.**

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APT approach delivers a solution!

Analytic Perturbation Theory

in

QCD

Analytic Perturb Theory (APT): Preamble

1st step: Improving PT by RG Method (Bogoliubov–Shirkov [1955-56]).

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(Bogoliubov–Logunov–Shirkov [1959], Radyushkin and Krasnikov&Pivovarov [1982]).

Its minimal (without extra parameters) version was devised by Jones&Solovtsov&Shirkov [1996–2006] and is known as Analytic Perturbation Theory.

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3rd step: Generalizing APT by including fractional powers of coupling and its products with logarithms due to principle of analytization "as a whole" (Karanikas–Stefanis, [2001]) in (A. B.&Mikhailov&Stefanis [2005–2009]) $\Rightarrow$ Fractional APT.

Basics of pQCD

- coupling $\alpha_s(\mu^2) = (4\pi/b_0)\, a_s[L]$ with $L = \ln(\mu^2/\Lambda^2)$
- RG equation $\dfrac{d\, a_s[L]}{d\, L} = -a_s^2 - c_1\, a_s^3 - \ldots$
- 1-loop solution generates Landau pole singularity: $a_s[L] = 1/L$
- 2-loop solution generates square-root singularity: $a_s[L] \sim 1/\sqrt{L + c_1 \ln c_1}$
- PT series: $D[L] = 1 + d_1 a_s[L] + d_2 a_s^2[L] + \ldots$

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- **Euclidean:** $-q^2 = Q^2,\ \ L = \ln Q^2/\Lambda^2,\ \ \{\mathcal{A}_n(L)\}_{n\in\mathbb{N}}$
- **Minkowskian:** $q^2 = s,\ \ L_s = \ln s/\Lambda^2,\ \ \{\mathfrak{A}_n(L_s)\}_{n\in\mathbb{N}}$

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  Causality $\Downarrow$ **Spectrality**

- Euclidean: $-q^2 = Q^2,\ \ L = \ln Q^2/\Lambda^2,\ \ \{\mathcal{A}_n(L)\}_{n\in\mathbb{N}}$
- Minkowskian: $q^2 = s,\ \ L_s = \ln s/\Lambda^2,\ \ \{\mathfrak{A}_n(L_s)\}_{n\in\mathbb{N}}$
- **PT** $\sum\limits_m d_m a_s^m(Q^2) \quad \Rightarrow \quad \sum\limits_m d_m \mathcal{A}_m(Q^2)$ **APT**

  m is power $\quad \Rightarrow \quad$ m is index

Spectral representation

By analytization we mean “Källen–Lehmann” representation

$$\left[f(Q^2)\right]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma$$

Then (note here pole remover):

$$\begin{aligned}
\rho(\sigma) &= \frac{1}{L_\sigma^2 + \pi^2} \\
\mathcal{A}_1[L] &= \int_0^\infty \frac{\rho(\sigma)}{\sigma + Q^2}\, d\sigma = \frac{1}{L} - \frac{1}{e^L - 1} \\
\mathfrak{A}_1[L_s] &= \int_s^\infty \frac{\rho(\sigma)}{\sigma}\, d\sigma = \frac{1}{\pi} \arccos \frac{L_s}{\sqrt{\pi^2 + L_s^2}}
\end{aligned}$$

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with spectral density $\rho_f(\sigma) = \textbf{Im}\left[f(-\sigma)\right]/\pi$. Then:

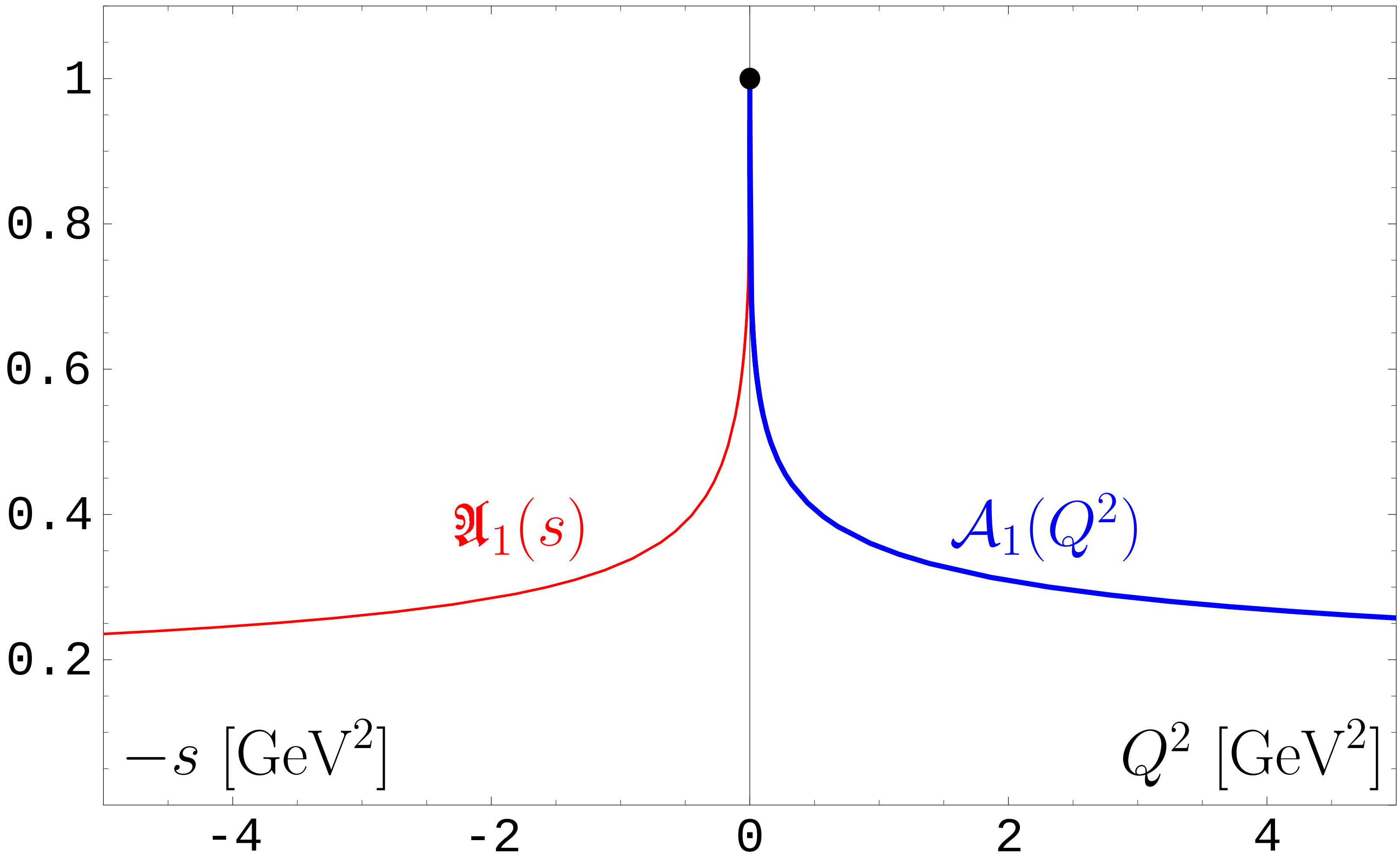
$$\mathcal{A}_n[L] = \int_0^\infty \frac{\rho_n(\sigma)}{\sigma + Q^2}\, d\sigma = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1}\mathcal{A}_1[L]$$

$$\mathfrak{A}_n[L_s] = \int_s^\infty \frac{\rho_n(\sigma)}{\sigma}\, d\sigma = \frac{1}{(n-1)!}\left(-\frac{d}{dL_s}\right)^{n-1}\mathfrak{A}_1[L_s]$$

$$a_s^n[L] = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1} a_s[L]$$

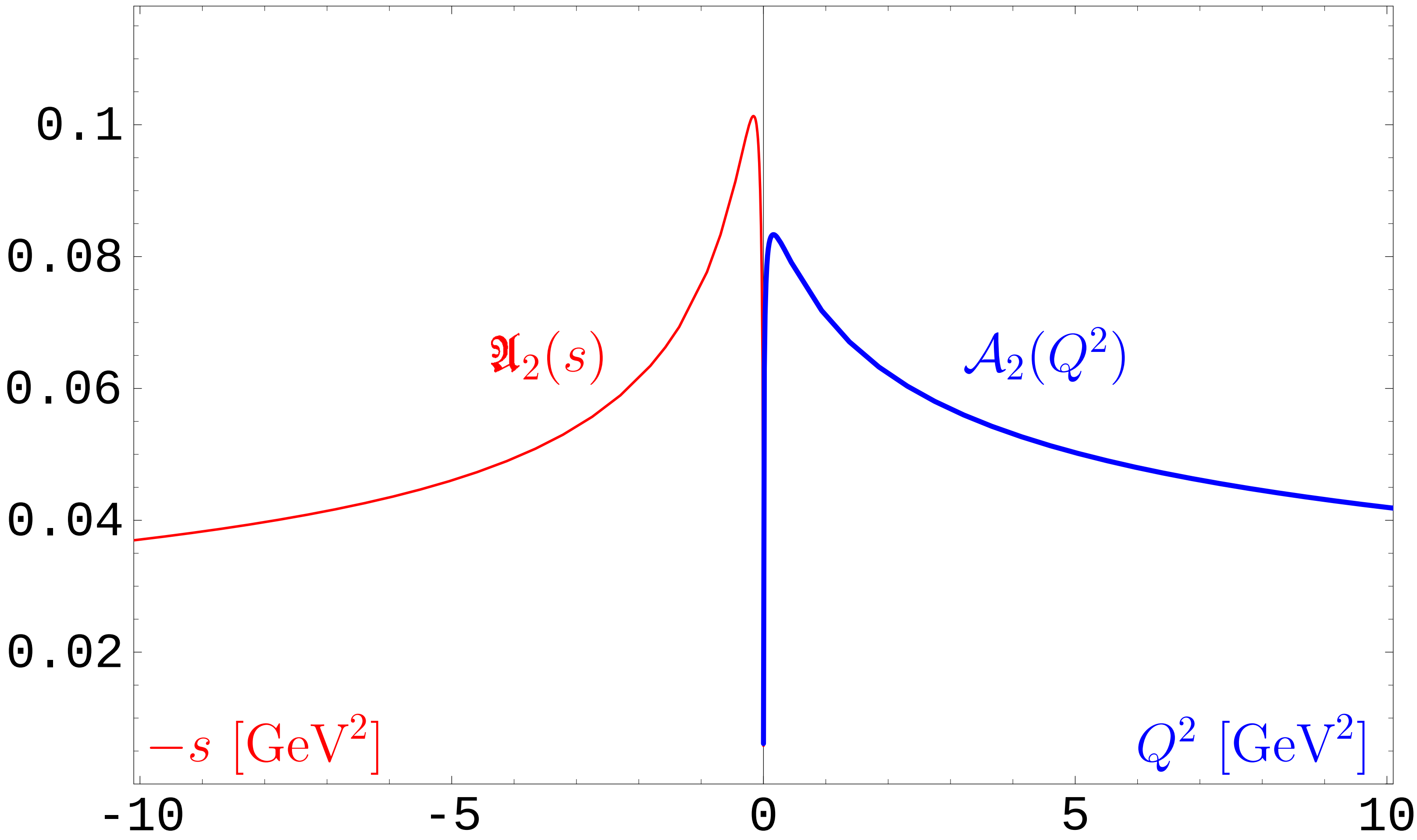
APT graphics: Distorting mirror

First, couplings: $\mathfrak{A}_1(s)$ **and** $\mathcal{A}_1(Q^2)$



APT graphics: Distorting mirror

Second, square-images: $\mathfrak{A}_2(s)$ **and** $\mathcal{A}_2(Q^2)$



Need
to use
Fractional APT

Problems of APT

In standard QCD PT we have not only power series $F[L] = \sum_m f_m\, a_s^m[L]$**, but also:**

- **Factorization** $\to (a_s[L])^n\, L^m$
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$$Z[L] = \exp\left\{\int^{a_s[L]} \frac{\gamma(a)}{\beta(a)} \, da\right\} \xrightarrow{\text{1-loop}} [a_s[L]]^{\gamma_0/(2\beta_0)}$$

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- **Two-loop case** $\to (a_s)^\nu \, \mathbf{ln}(a_s)$

New functions: $(a_s)^\nu\,,\ (a_s)^\nu \, \mathbf{ln}(a_s),\ (a_s)^\nu \, L^m,\ \ldots$

Constructing one-loop FAPT

In one-loop APT we have a very nice recurrence relation

$$\mathcal{A}_n[L] = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1}\mathcal{A}_1[L]$$

and the same in Minkowski domain

$$\mathfrak{A}_n[L] = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1}\mathfrak{A}_1[L]\,.$$

We can use it to construct FAPT .

FAPT(E): Properties of $\mathcal{A}_\nu[L]$

First, Euclidean coupling ($L = L(Q^2)$):

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}$$

Here $F(z,\nu)$ is reduced Lerch transcendent. function. It is analytic function in ν.

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- $\mathcal{A}_0[L] = 1$;
- $\mathcal{A}_{-m}[L] = L^m$ for $m \in \mathbb{N}$;
- $\mathcal{A}_m[L] = (-1)^m \mathcal{A}_m[-L]$ for $m \geq 2\,,\ m \in \mathbb{N}$;
- $\mathcal{A}_m[\pm\infty] = 0$ for $m \geq 2\,,\ m \in \mathbb{N}$;

FAPT(M): Properties of $\mathfrak{A}_\nu[L]$

Now, Minkowskian coupling ($L = L(s)$**):**

$$\mathfrak{A}_\nu[L] = \frac{\sin\left[(\nu-1)\text{arccos}\left(L/\sqrt{\pi^2+L^2}\right)\right]}{\pi(\nu-1)\left(\pi^2+L^2\right)^{(\nu-1)/2}}$$

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- $\mathfrak{A}_0[L] = 1$;
- $\mathfrak{A}_{-1}[L] = L$;
- $\mathfrak{A}_{-2}[L] = L^2 - \dfrac{\pi^2}{3}, \quad \mathfrak{A}_{-3}[L] = L\left(L^2 - \pi^2\right), \ \ldots$;
- $\mathfrak{A}_m[L] = (-1)^m \mathfrak{A}_m[-L]$ **for** $m \geq 2\,,\ m \in \mathbb{N}$;
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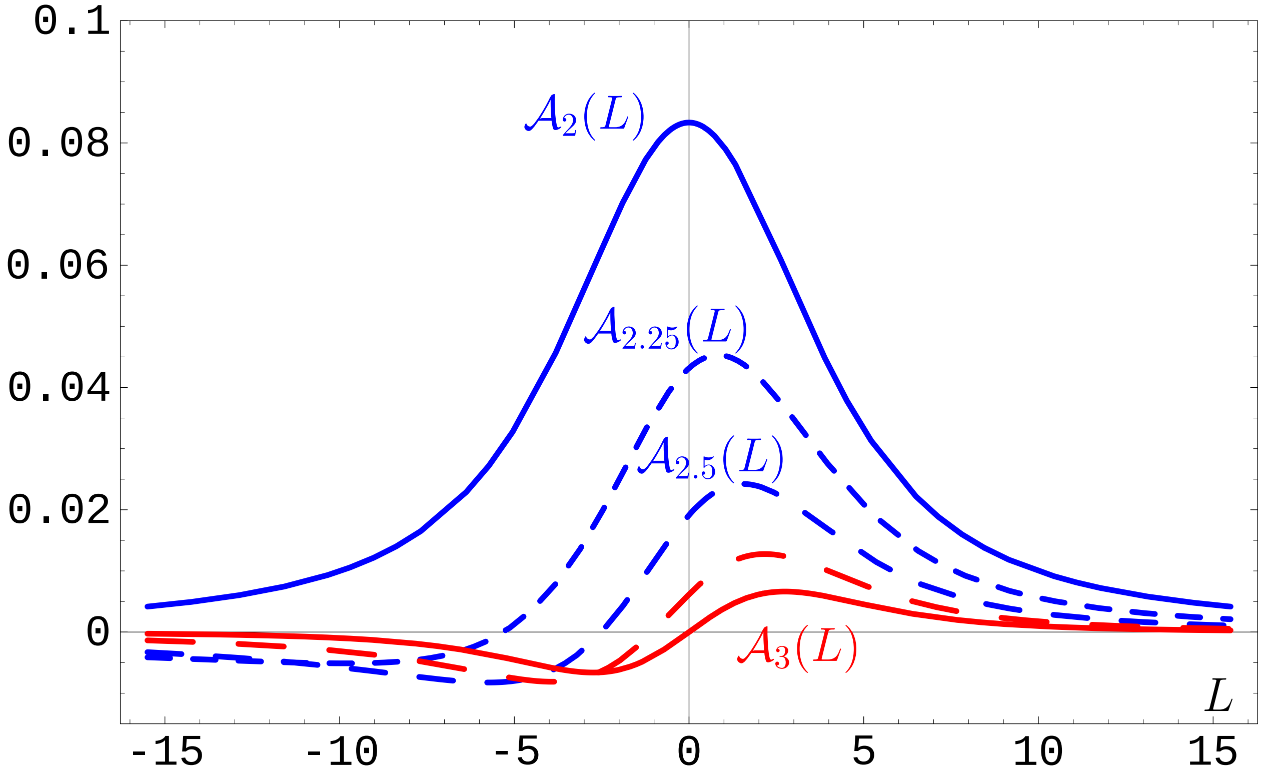
FAPT(E): Graphics of $\mathcal{A}_\nu[L]$ vs. L

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}$$

Graphics for fractional $\nu \in [2, 3]$:



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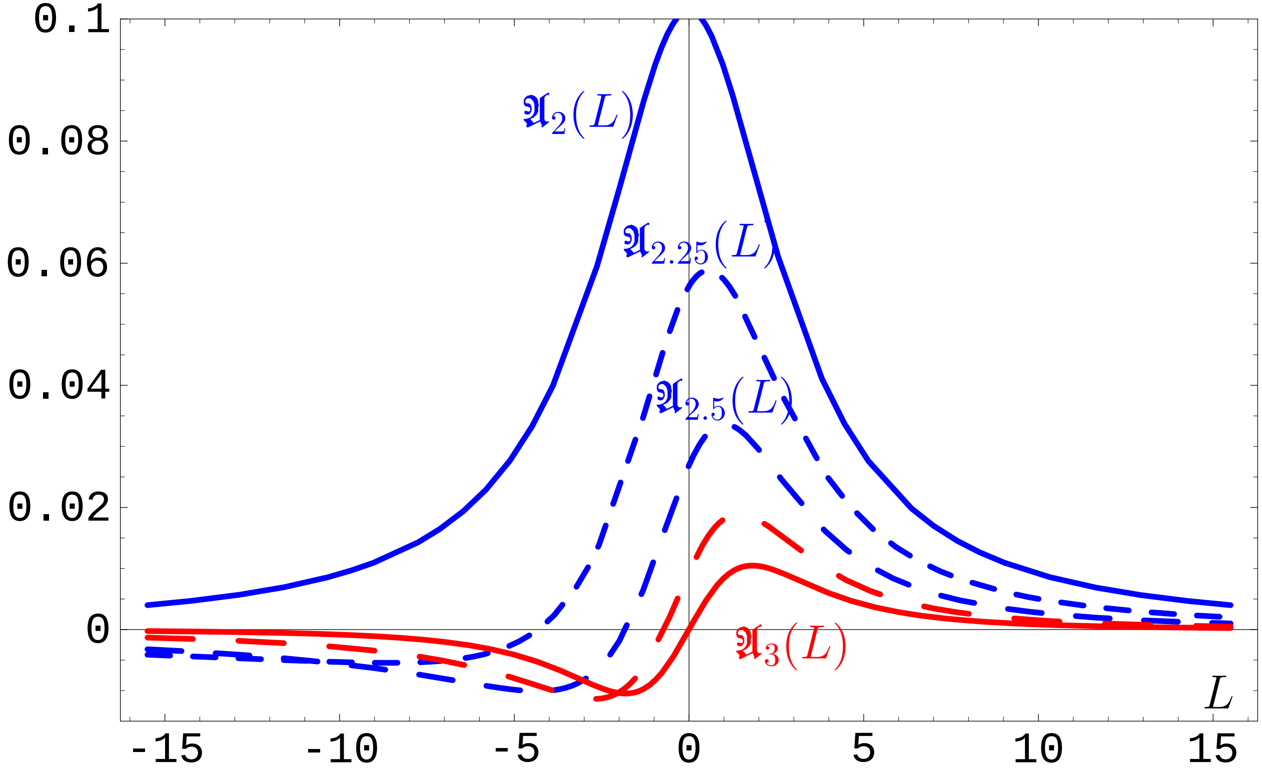
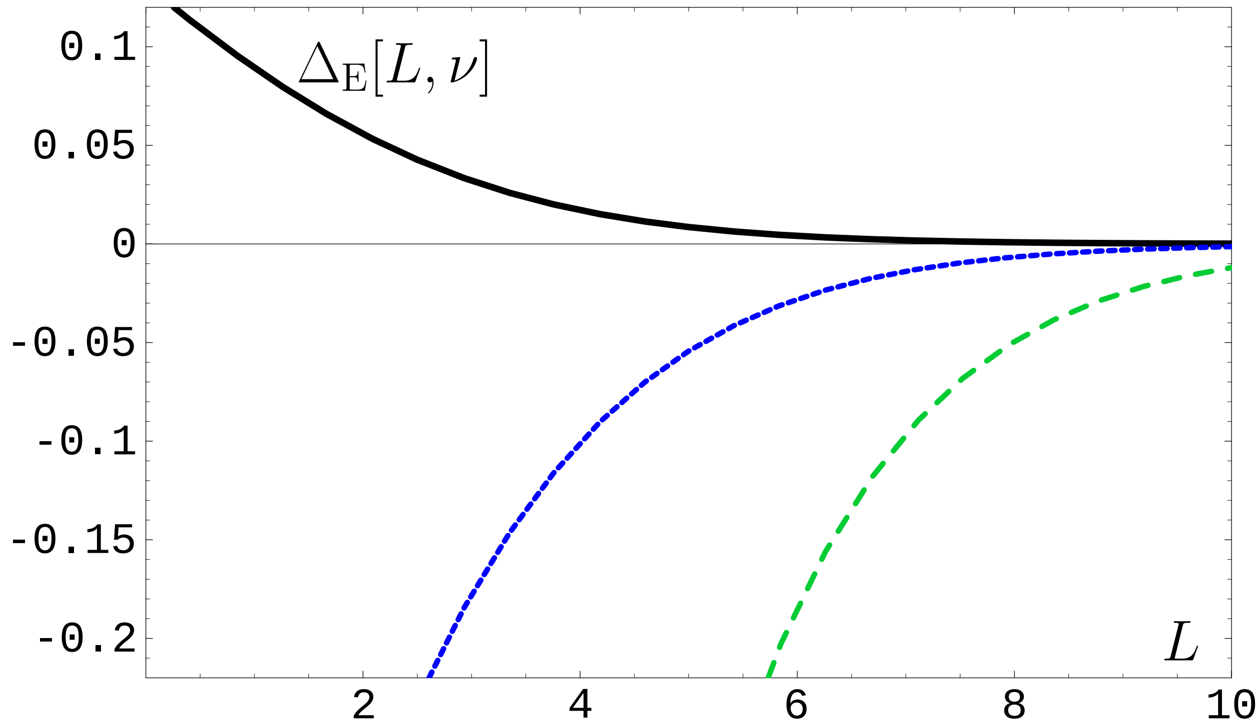
Compare with graphics in Minkowskian region :



FAPT(E): Comparing $\mathcal{A}_\nu$ with $(\mathcal{A}_1)^\nu$

$$\Delta_{\mathrm{E}}(L,\nu) = \frac{\mathcal{A}_\nu[L] - (\mathcal{A}_1[L])^\nu}{\mathcal{A}_\nu[L]}$$

Graphics for fractional $\nu =$0.62, 1.62 and 2.62:



The larger ν is — the more important FAPT becomes!

FAPT(M): Comparing $\mathfrak{A}_\nu$ with $\left(\mathfrak{A}_1\right)^\nu$

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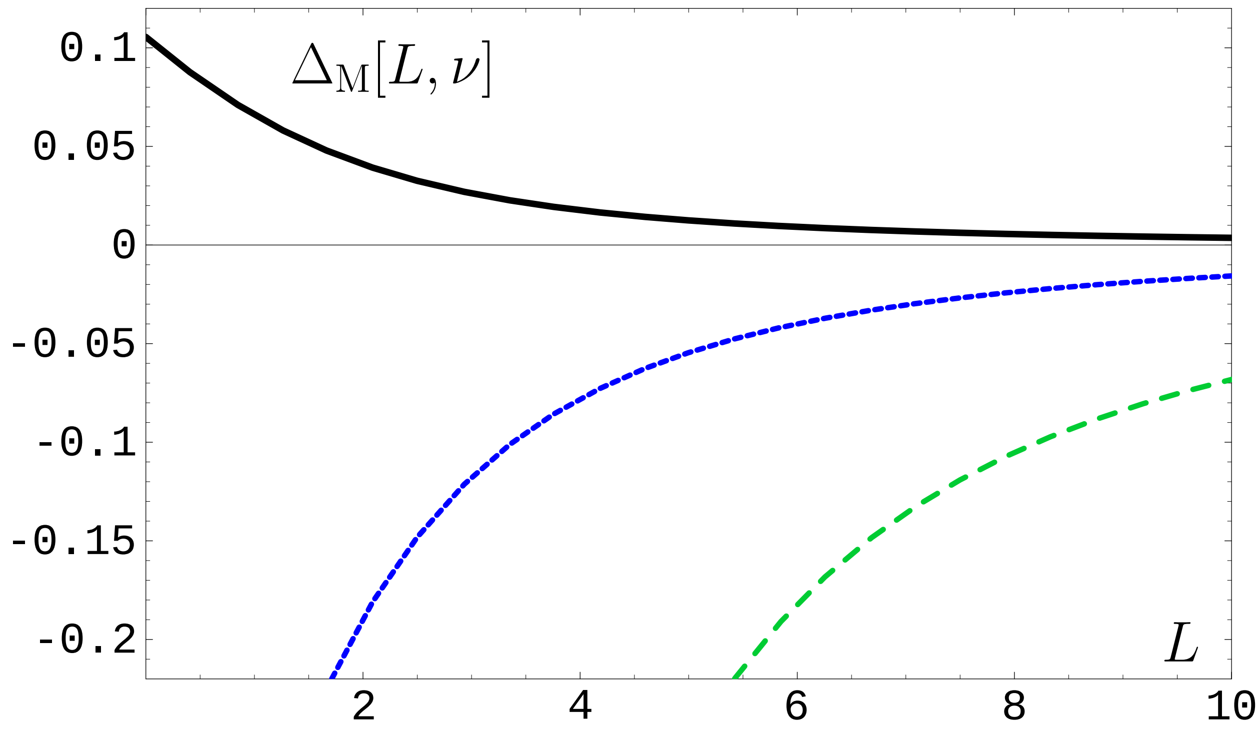
Minkowskian graphics for $\nu =$0.62, 1.62 and 2.62:



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Resummation in one-loop APT and FAPT

Generating function for coefficients

Consider series $\mathcal{D}[L] = d_0 + \sum_{n=1}^{\infty} d_n \, \mathcal{A}_n[L]$

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Let exists the generating function $P(t)$ for coefficients:

$$d_n = d_1 \int_0^{\infty} P(t)\, t^{n-1} dt \quad \text{with} \quad \int_0^{\infty} P(t)\, dt = 1\,.$$

We define a shorthand notation

$$\langle\langle f(t) \rangle\rangle_{P(t)} \equiv \int_0^{\infty} f(t)\, P(t)\, dt\,.$$

Then coefficients $d_n = d_1 \,\langle\langle t^{n-1} \rangle\rangle_{P(t)}$.

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and for Minkowski region:

$$\mathcal{R}[L] = d_0 + d_1 \,\langle\langle \mathfrak{A}_1[L-t] \rangle\rangle_{P(t)}$$

Models for perturbative coefficients

Coefficients d_n of the PT series:

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$(t/c)^{\gamma+1}e^{-t/c}$	$n^{\gamma}\,c^n\Gamma(n+1)$

Resummation in one-loop FAPT

Consider series $\mathcal{R}_\nu[L] = d_0\,\mathfrak{A}_\nu[L] + \sum_{n=1}^{\infty} d_n\,\mathfrak{A}_{n+\nu}[L]$

or $\mathcal{D}_\nu[L] = d_0\,\mathcal{A}_\nu[L] + \sum_{n=1}^{\infty} d_n\,\mathcal{A}_{n+\nu}[L]$

with coefficients $d_n = d_1\,\langle\langle t^{n-1}\rangle\rangle_{P(t)}$.

Result:

$$\begin{aligned}
\mathcal{R}_\nu[L] &= d_0\,\mathfrak{A}_\nu[L] + d_1\,\langle\langle\mathfrak{A}_{1+\nu}[L-t]\rangle\rangle_{P_\nu(t)}\,;\\
\mathcal{D}_\nu[L] &= d_0\,\mathcal{A}_\nu[L] + d_1\,\langle\langle\mathcal{A}_{1+\nu}[L-t]\rangle\rangle_{P_\nu(t)}\,.
\end{aligned}$$

where $P_\nu(t) = \int_0^1 P\left(\frac{t}{1-z}\right)\nu\, z^{\nu-1}\frac{dz}{1-z}\,.$

Resummation
in
two- and three-loop FAPT

Resummation in two-loop FAPT

Consider series $\mathcal{S}_\nu[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \mathcal{F}_{n+\nu}[L]$.

Here $\mathcal{F}_\nu[L] = \mathcal{A}_\nu^{(2)}[L]$ or $\mathfrak{A}_\nu^{(2)}[L]$ (or $\rho_\nu^{(2)}[L]$ — for global).

Resummation in two-loop FAPT

Consider series $\mathcal{S}_\nu[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \, \mathcal{F}_{n+\nu}[L]$.

Here $\mathcal{F}_\nu[L] = \mathcal{A}_\nu^{(2)}[L]$ **or** $\mathfrak{A}_\nu^{(2)}[L]$ **(or** $\rho_\nu^{(2)}[L]$ **— for global).**

We have two-loop recurrence relation ($c_1 = b_1/b_0^2$**):**

$$-\frac{1}{n+\nu}\,\frac{d}{dL}\,\mathcal{F}_{n+\nu}[L] = \mathcal{F}_{n+1+\nu}[L] + c_1\,\mathcal{F}_{n+2+\nu}[L]\,.$$

Resummation in two-loop FAPT

Consider series $\mathcal{S}_\nu[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \mathcal{F}_{n+\nu}[L]$.

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We have two-loop recurrence relation ($c_1 = b_1/b_0^2$**):**

$$-\frac{1}{n+\nu}\frac{d}{dL}\mathcal{F}_{n+\nu}[L] = \mathcal{F}_{n+1+\nu}[L] + c_1\,\mathcal{F}_{n+2+\nu}[L]\,.$$

In order to resum our series we need to define the two-loop time $\tau_2(t) = t - c_1 \ln\left[1 + \frac{t}{c_1}\right]$ **with** $\frac{d\tau_2(t)}{dt} = \frac{1}{1 + c_1/t}$

to be compared with standard two-loop evolution time $\tau_{(2)}(t)$

with $\frac{dt}{d\tau_{(2)}(t)} = \frac{1}{1 + c_1/\tau_{(2)}(t)}$

Resummation in two-loop FAPT

Consider series $\mathcal{S}_\nu[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)}\, \mathcal{F}_{n+\nu}[L]$.

Here $\mathcal{F}_\nu[L] = \mathcal{A}_\nu^{(2)}[L]$ or $\mathfrak{A}_\nu^{(2)}[L]$ (or $\rho_\nu^{(2)}[L]$ — for global).

We have two-loop recurrence relation ($c_1 = b_1/b_0^2$):

$$-\frac{1}{n+\nu}\,\frac{d}{dL}\,\mathcal{F}_{n+\nu}[L] = \mathcal{F}_{n+1+\nu}[L] + c_1\,\mathcal{F}_{n+2+\nu}[L]\,.$$

Result (with $\tau_2(t) = t - c_1 \ln(1 + t/c_1)$):

$$\mathcal{S}[L] = \left\langle\!\!\left\langle \mathcal{F}_{1+\nu}[L] - \frac{t^2}{c_1+t} \int_0^1 z^\nu dz\, \dot{\mathcal{F}}_{1+\nu}[L+\tau_2(t\,z)-\tau_2(t)] \right.\right.$$

$$\left.\left. + \frac{c_1\,t}{c_1+t} \left\{ \mathcal{F}_{2+\nu}[L] - \int_0^1 dz\, \frac{t^2\, z^{\nu+1}}{c_1+t\,z}\, \dot{\mathcal{F}}_{2+\nu}[L+\tau_2(t\,z)-\tau_2(t)] \right\} \right\rangle\!\!\right\rangle_{P(t)}$$

Resummation in three-loop FAPT

Consider series $\mathcal{S}_\nu[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \mathcal{F}_{n+\nu}[L]$.

Here $\mathcal{F}_\nu[L] = \mathcal{A}_\nu^{(2)}[L]$ **or** $\mathfrak{A}_\nu^{(2)}[L]$ **(or** $\rho_\nu^{(2)}[L]$ **— for global).**

Resummation in three-loop FAPT

Consider series $\mathcal{S}_\nu[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \, \mathcal{F}_{n+\nu}[L]$.

Here $\mathcal{F}_\nu[L] = \mathcal{A}_\nu^{(2)}[L]$ **or** $\mathfrak{A}_\nu^{(2)}[L]$ **(or** $\rho_\nu^{(2)}[L]$ **— for global).**

We have three-loop recurrence relation ($c_2 = b_2/b_0^3$**):**

$$\frac{-\,d\mathcal{F}_{n+\nu}[L]}{(n+\nu)\,dL} = \mathcal{F}_{n+1+\nu}[L] + c_1\,\mathcal{F}_{n+2+\nu}[L] + c_2\,\mathcal{F}_{n+3+\nu}[L]\,.$$

Resummation in three-loop FAPT

Consider series $\mathcal{S}_\nu[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \, \mathcal{F}_{n+\nu}[L]$.

Here $\mathcal{F}_\nu[L] = \mathcal{A}_\nu^{(2)}[L]$ or $\mathfrak{A}_\nu^{(2)}[L]$ (or $\rho_\nu^{(2)}[L]$ — for global).

We have three-loop recurrence relation ($c_2 = b_2/b_0^3$):

$$\frac{-\, d\mathcal{F}_{n+\nu}[L]}{(n+\nu)\, dL} = \mathcal{F}_{n+1+\nu}[L] + c_1 \, \mathcal{F}_{n+2+\nu}[L] + c_2 \, \mathcal{F}_{n+3+\nu}[L] \, .$$

Now, to resum our series, we need to define the three-loop time $\tau_3(t)$ with $\dfrac{d\tau_3(t)}{dt} = \dfrac{1}{1 + (c_1/t) + c_2/t^2}$

to be compared with standard three-loop evolution time $\tau_{(3)}(t)$

with $\dfrac{dt}{d\tau_{(3)}(t)} = \dfrac{1}{1 + (c_1/\tau_{(3)}(t)) + c_2/\tau_{(3)}(t)^2}$

Resummation in three-loop FAPT

Consider series $\mathcal{S}_\nu[L] = \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \, \mathcal{F}_{n+\nu}[L]$.

Here $\mathcal{F}_\nu[L] = \mathcal{A}_\nu^{(2)}[L]$ or $\mathfrak{A}_\nu^{(2)}[L]$ (or $\rho_\nu^{(2)}[L]$ — for global).
We have three-loop recurrence relation ($c_2 = b_2/b_0^3$):

$$\frac{-\, d\mathcal{F}_{n+\nu}[L]}{(n+\nu)\, dL} = \mathcal{F}_{n+1+\nu}[L] + c_1 \, \mathcal{F}_{n+2+\nu}[L] + c_2 \, \mathcal{F}_{n+3+\nu}[L]\,.$$

Result ($L_{z,t} \equiv L + \tau_3(t\,z) - \tau_3(t)$]):

$$\mathcal{S}[L] = \left\langle\!\!\left\langle \mathcal{F}_{1+\nu}[L] + t\, \mathcal{F}_{2+\nu}[L] - \frac{t^2}{t^2 + c_1\, t + c_2} \int_0^1 z^\nu dz \left\{ t\, \dot{\mathcal{F}}_{1+\nu}[L_{z,t}] \right.\right.\right.$$

$$\left.\left.\left. + \, z\, t^2 \dot{\mathcal{F}}_{2+\nu}[Lz,t] + (\nu+1)\, t\, \mathcal{F}_{2+\nu}[Lz,t] - \frac{c_2\, \nu}{z} \mathcal{F}_{3+\nu}[Lz,t] \right\} \right\rangle\!\!\right\rangle_{P(t)}$$

Resummation for Adler function $D(Q^2)$

Adler function $D(Q^2)$ in vector channel

Adler function $D(Q^2)$ can be expressed in QCD by means of the correlator of quark vector currents

$$\Pi_{\mathsf{V}}(Q^2) = \frac{(4\pi)^2}{3q^2}\, i \int dx\, e^{iqx} \langle 0|\, T[\, J_\mu(x) J^\mu(0)\,]\, |0\rangle$$

in terms of discontinuity of its imaginary part

$$R_{\mathsf{V}}(s) = \frac{1}{\pi}\, \mathrm{Im}\, \Pi_{\mathsf{V}}(-s - i\epsilon)\,,$$

so that

$$D(Q^2) = Q^2 \int_0^\infty \frac{R_{\mathsf{V}}(\sigma)}{(\sigma + Q^2)^2}\, d\sigma\,.$$

APT analysis of $D(Q^2)$ *and* $R_V(s)$

QCD PT gives us

$$D(Q^2) = 1 + \sum_{m>0} \frac{d_m}{\pi^m} \left(\alpha_s(Q^2)\right)^m .$$

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$$D(Q^2) = 1 + \sum_{m>0} \frac{d_m}{\pi^m} \left(\alpha_s(Q^2)\right)^m .$$

In APT (E) we obtain

$$\mathcal{D}_N(Q^2) = 1 + \sum_{m>0}^{N} \frac{d_m}{\pi^m} \mathcal{A}_m^{\textbf{glob}}(Q^2)$$

APT analysis of $D(Q^2)$ *and* $R_V(s)$

QCD PT gives us

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In APT (E) we obtain

$$\mathcal{D}_N(Q^2) = 1 + \sum_{m>0}^{N} \frac{d_m}{\pi^m} \mathcal{A}_m^{\text{glob}}(Q^2)$$

and in APT (M)

$$\mathcal{R}_{\text{V};N}(s) = 1 + \sum_{m>0}^{N} \frac{d_m}{\pi^m} \mathfrak{A}_m^{\text{glob}}(s)$$

Model for perturbative coefficients

Coefficients d_m of the PT series:

Model	d_1	d_2	d_3	d_4	d_5
pQCD with $N_f = 4$	$\mathbf{1}$	$\mathbf{1.52}$	$\mathbf{2.59}$		—

Model for perturbative coefficients

Coefficients d_m of the PT series:

Model	d_1	d_2	d_3	d_4	d_5
pQCD with $N_f = 4$	1	1.52	2.59		—
$c = 3.467,\ \beta = 1.325$	1	1.50	2.62		

We use model $\tilde{d}_n^{\text{mod}} = \dfrac{c^{n-1}(\beta^{n+1} - n)}{\beta^2 - 1}\,\Gamma(n)$

with parameters β and c estimated by known $\tilde{d}_n$

that possesses the Lipatov asymptotics $\tilde{d}_n^{\text{mod}} \sim b^n n!$ at $n \gg 1$.

Model for perturbative coefficients

Coefficients d_m of the PT series:

Model	d_1	d_2	d_3	d_4	d_5
pQCD with $N_f = 4$	1	1.52	2.59	27.4	—
$c = 3.467,\ \beta = 1.325$	1	1.50	2.62	27.8	

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Improving the parameters — like in Kalman algorithm.

Model for perturbative coefficients

Coefficients d_m of the PT series:

Model	d_1	d_2	d_3	d_4	d_5
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$c = 3.467,\ \beta = 1.325$	1	1.50	2.62	27.8	
$c = 3.456,\ \beta = 1.325$	1	1.49	2.60	27.5	

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"INNA" model	1	1.44	$[3,9]$	$[20,48]$	$[674,2786]$

We use model $\tilde{d}_n^{\text{mod}} = \dfrac{c^{n-1}(\beta^{n+1} - n)}{\beta^2 - 1}\,\Gamma(n)$

with parameters β and c estimated by known $\tilde{d}_n$

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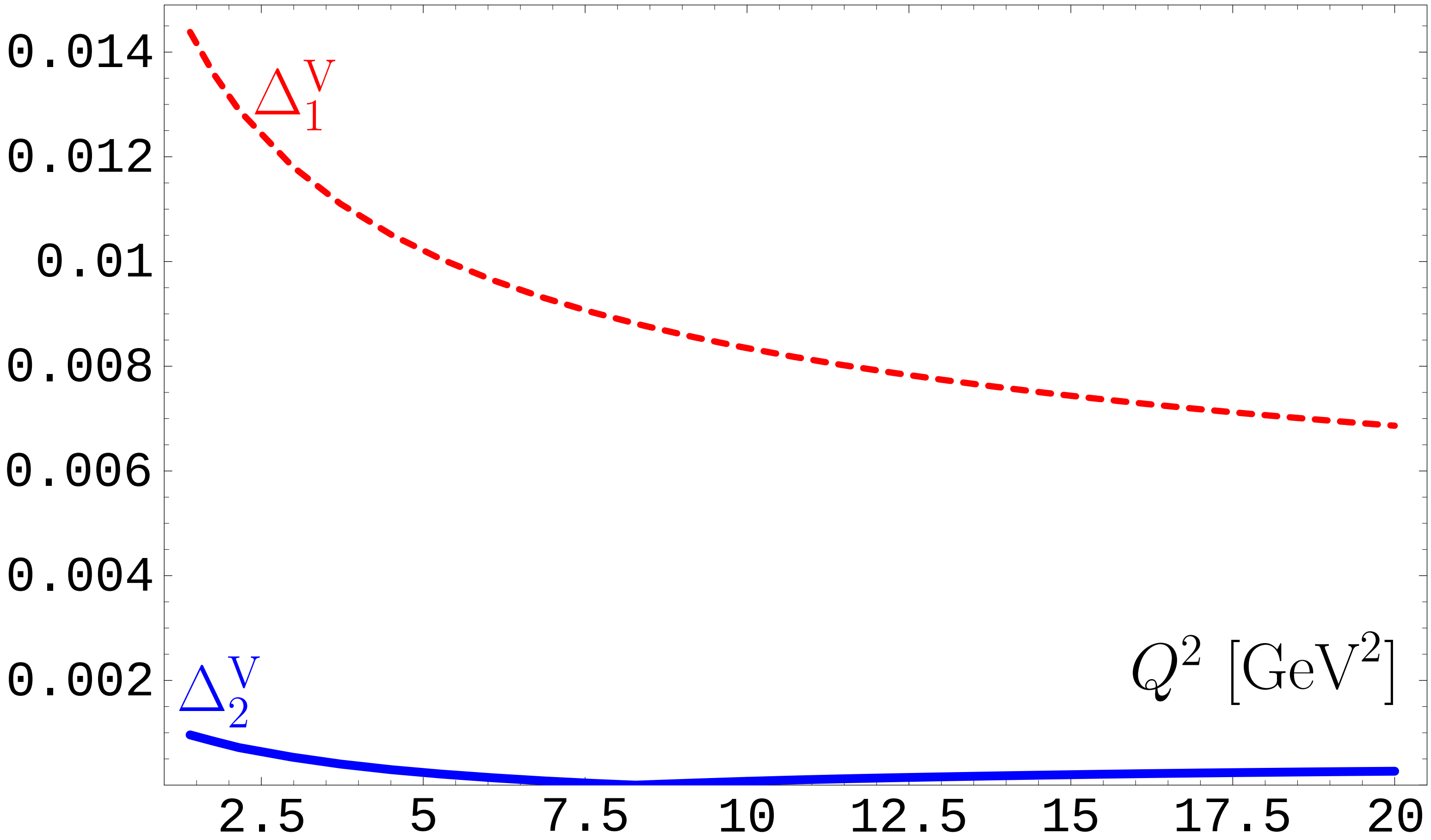
One-loop APT(E) for $\mathcal{D}(Q^2)$: Truncation errors

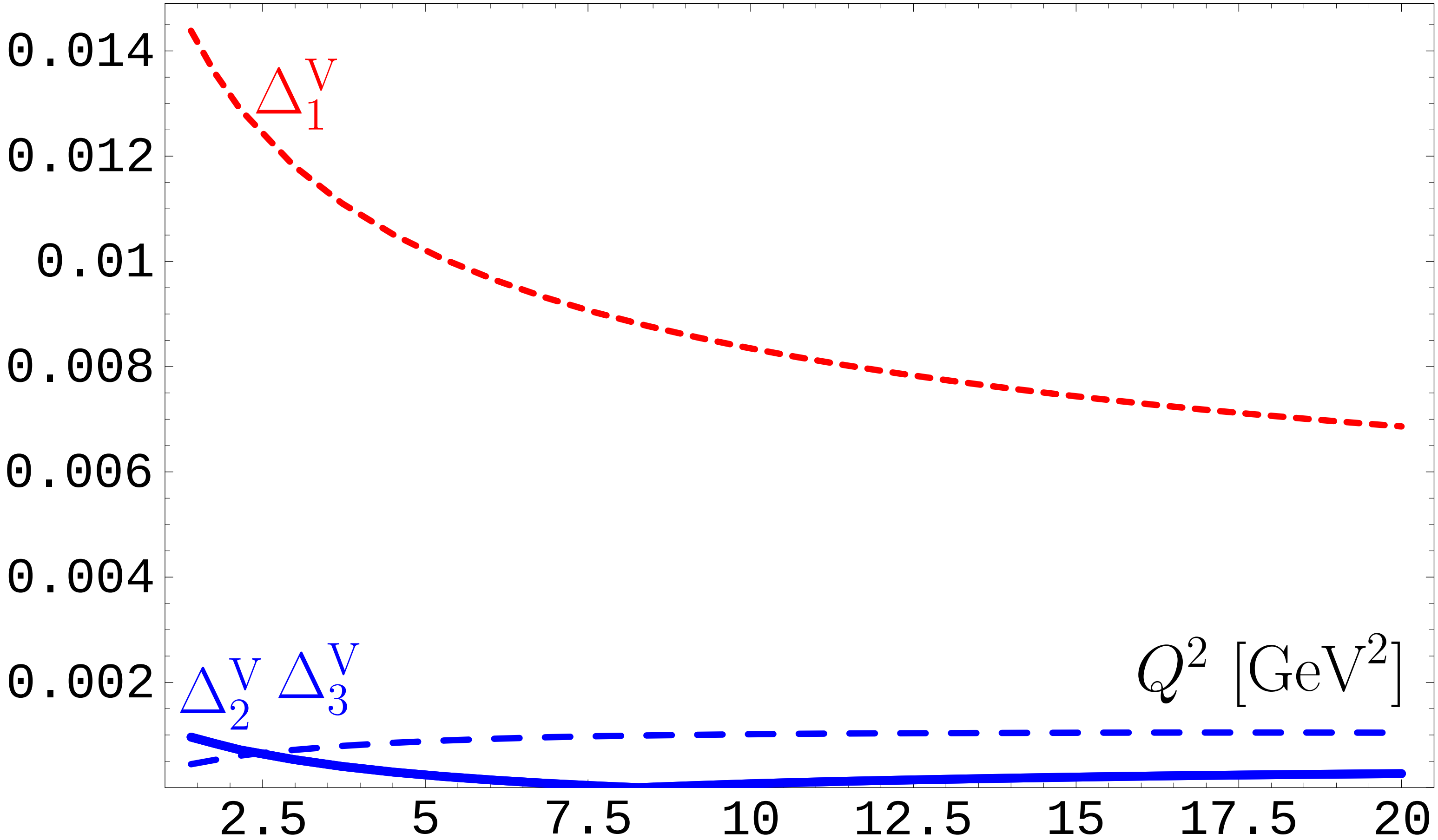
We define relative errors of series truncation at Nth term:

$$\Delta_N^{\mathrm{V}}[L] = 1 - \mathcal{D}_N[L]/\mathcal{D}_\infty[L]$$



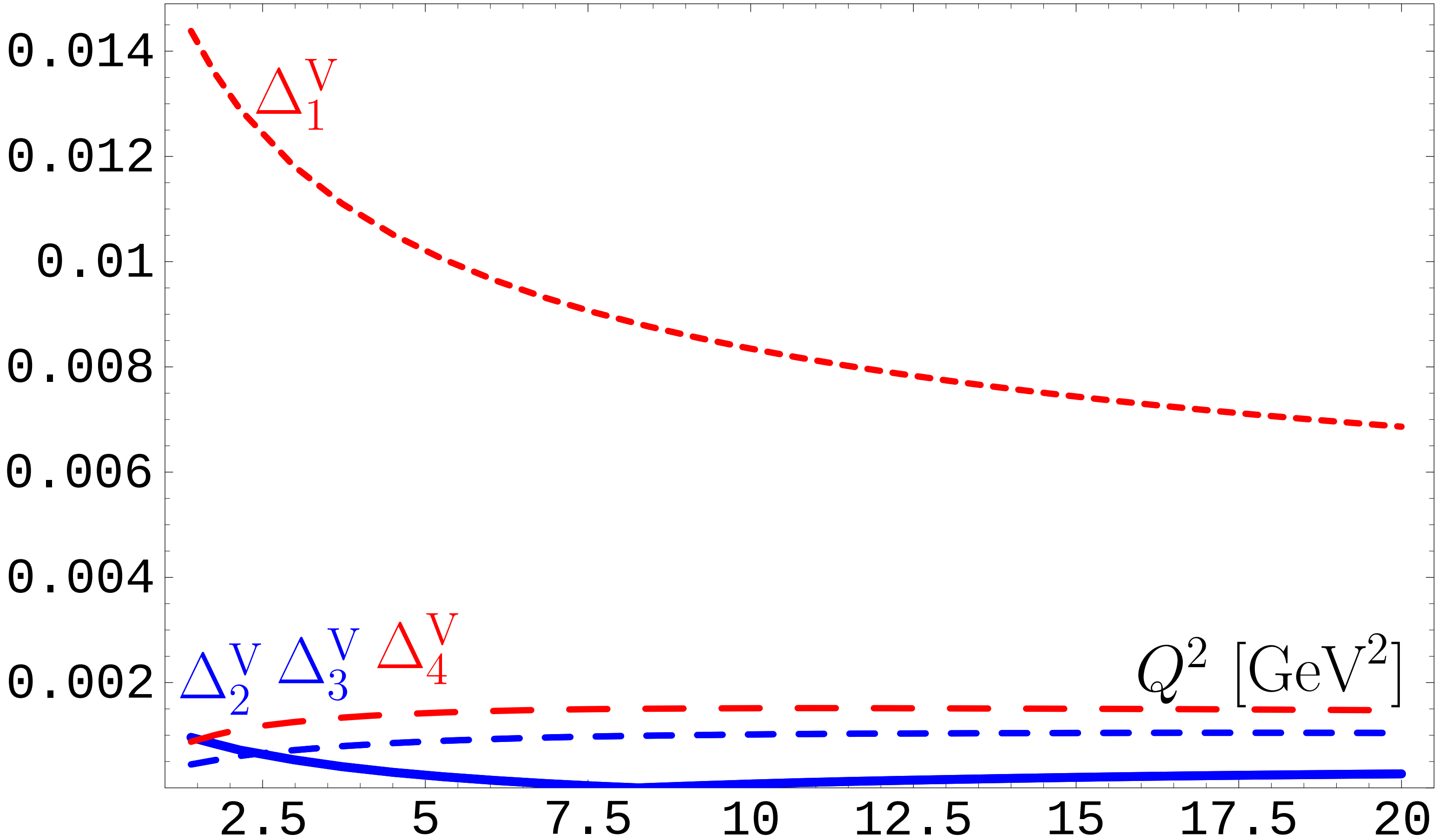
One-loop APT(E) for $\mathcal{D}(Q^2)$: Truncation errors

Conclusion: **The best accuracy (better than 0.1%) is achieved for N^{2}LO approximation.**



One-loop APT(E) for $\mathcal{D}(Q^2)$: Truncation errors

Conclusion: If we add more terms N^{3}LO — truncation error increases.



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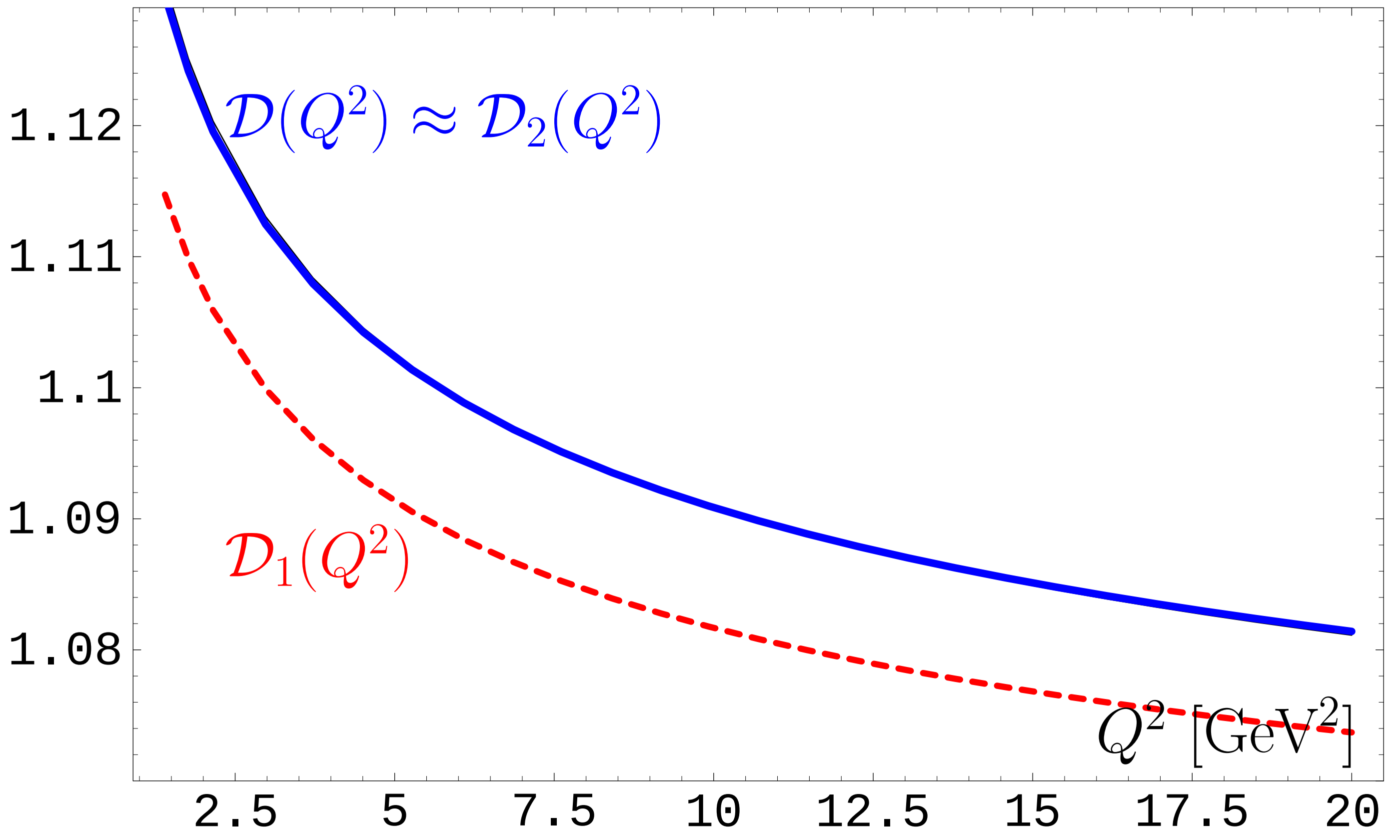

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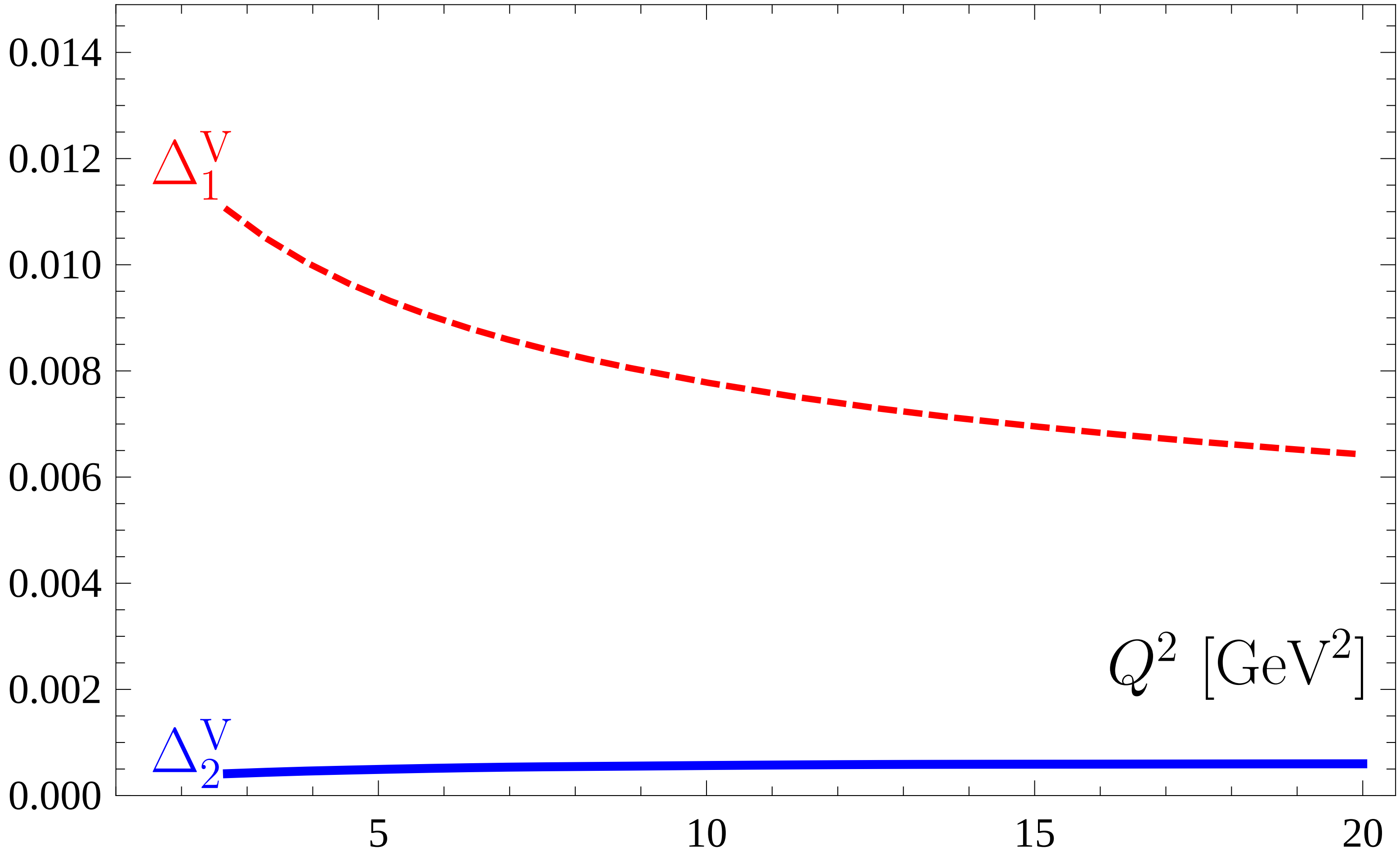

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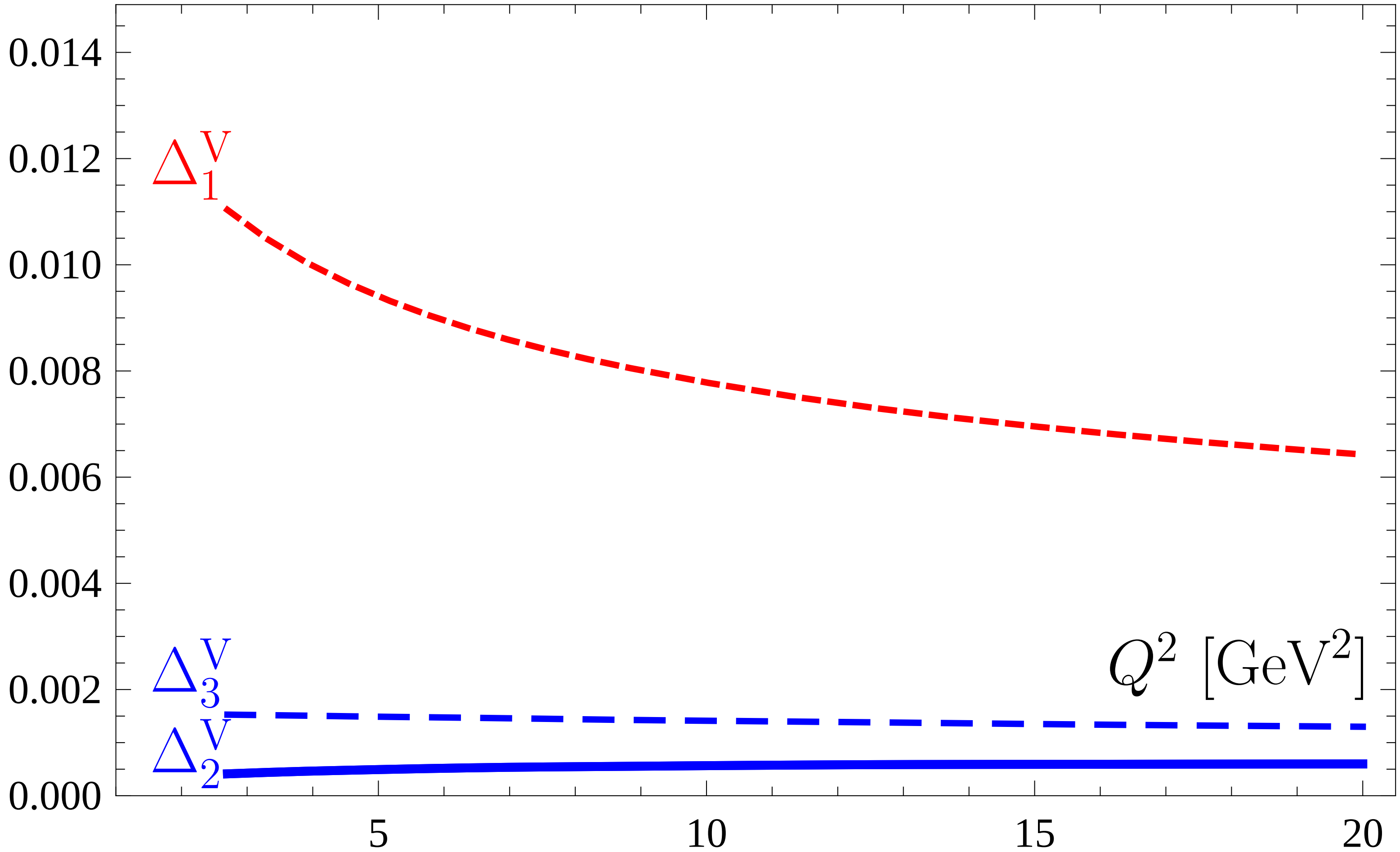

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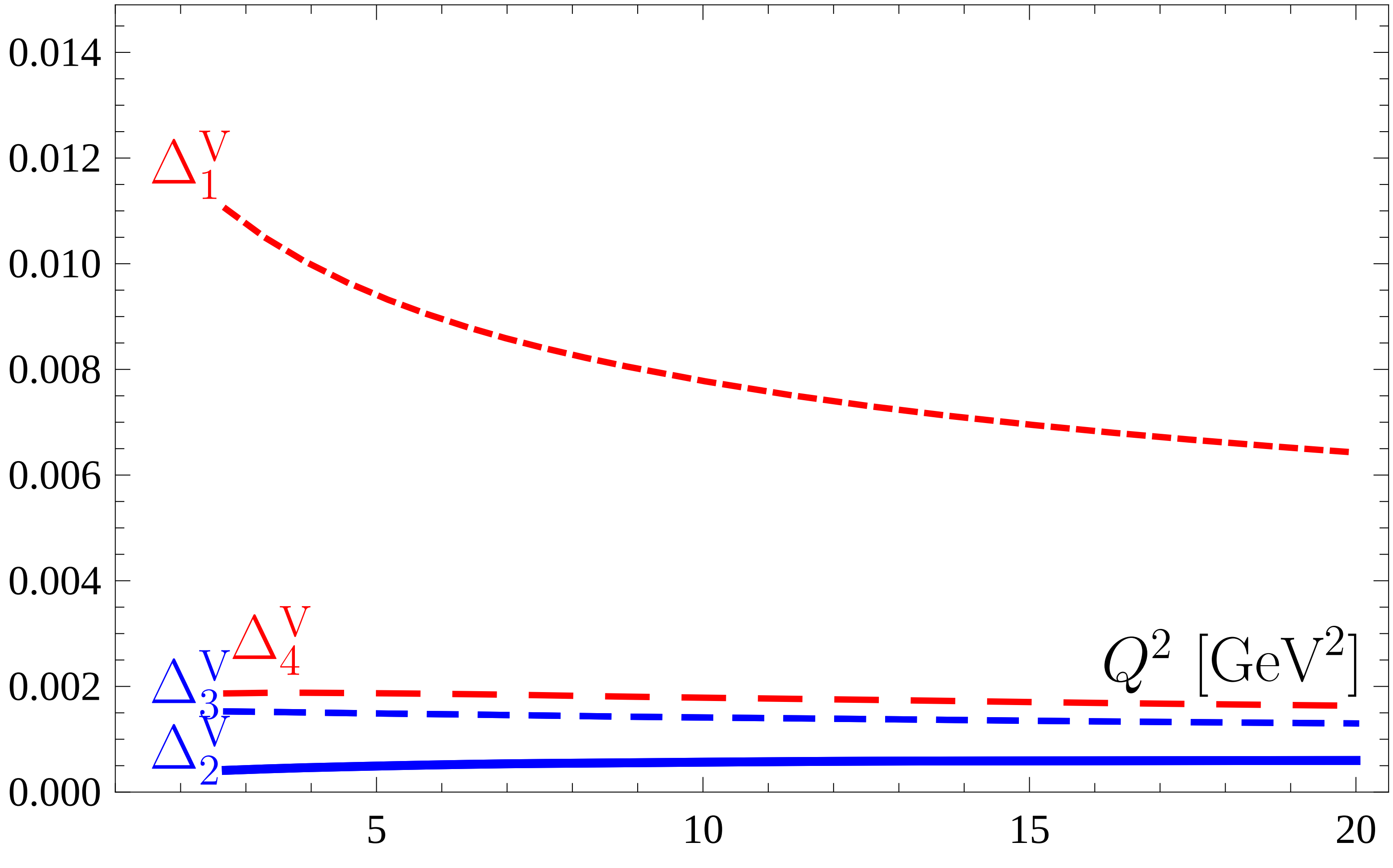

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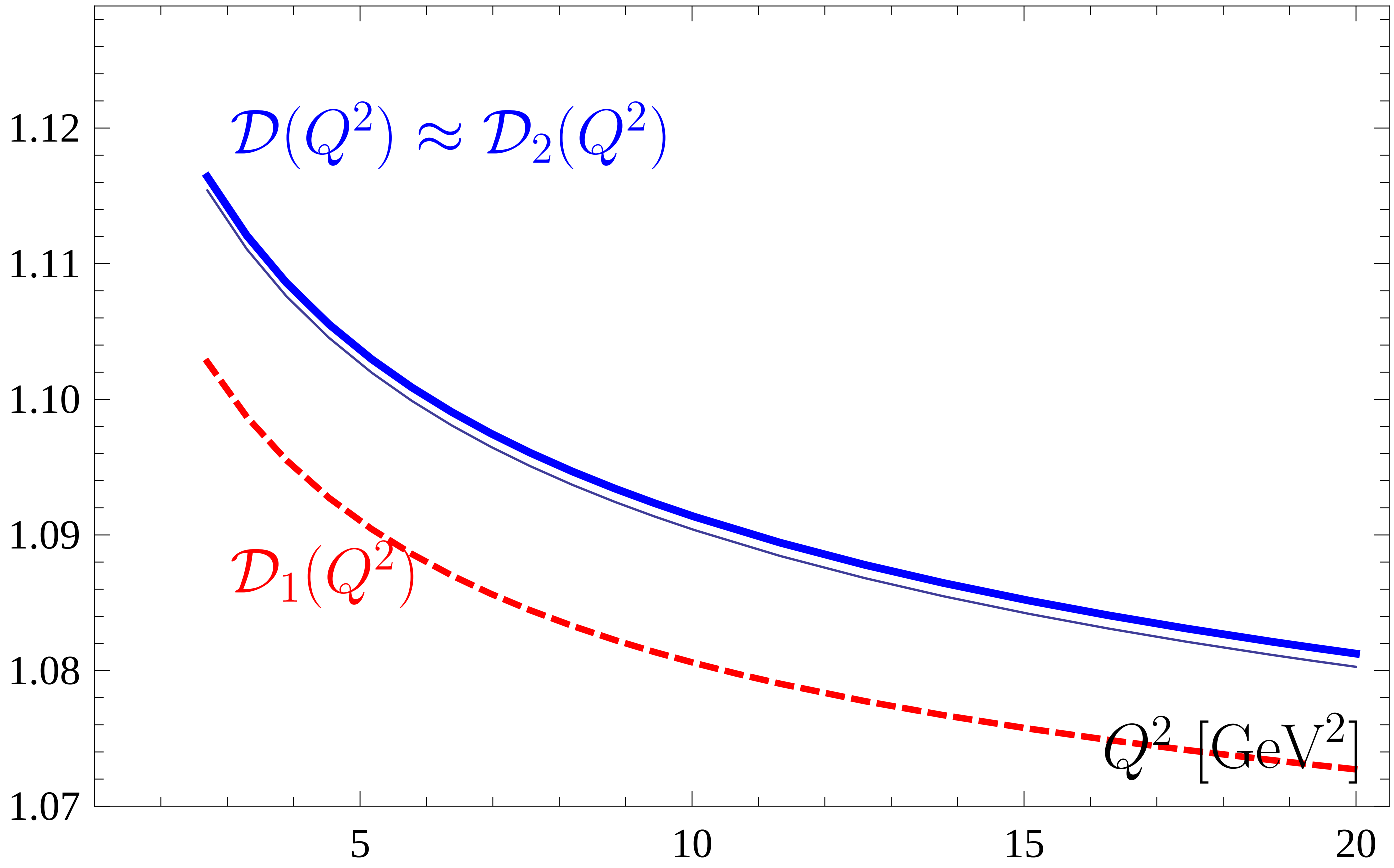

APT(E) for $\mathcal{D}(Q^2)$: Errors of modelling $P(t)$

We use model $d_n^{\text{mod}} = \dfrac{c^{n-1}(\beta^{n+1} - n)}{\beta^2 - 1}\,\Gamma(n)$ with parameters $\beta = 1.325$ and $c = 3.456$ estimated by known $\tilde{d}_n$ and with use of Lipatov asymptotics.

We apply it to resum APT series and obtain $\mathcal{D}(Q^2)$.

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We deform our model for d_n by using coefficients $\beta_{\text{NNA}} = 1.322$ and $c_{\text{NNA}} = 3.885$

that deforms $d_4 = 27.5 \to d_4^{\text{NNA}} = 20.4$

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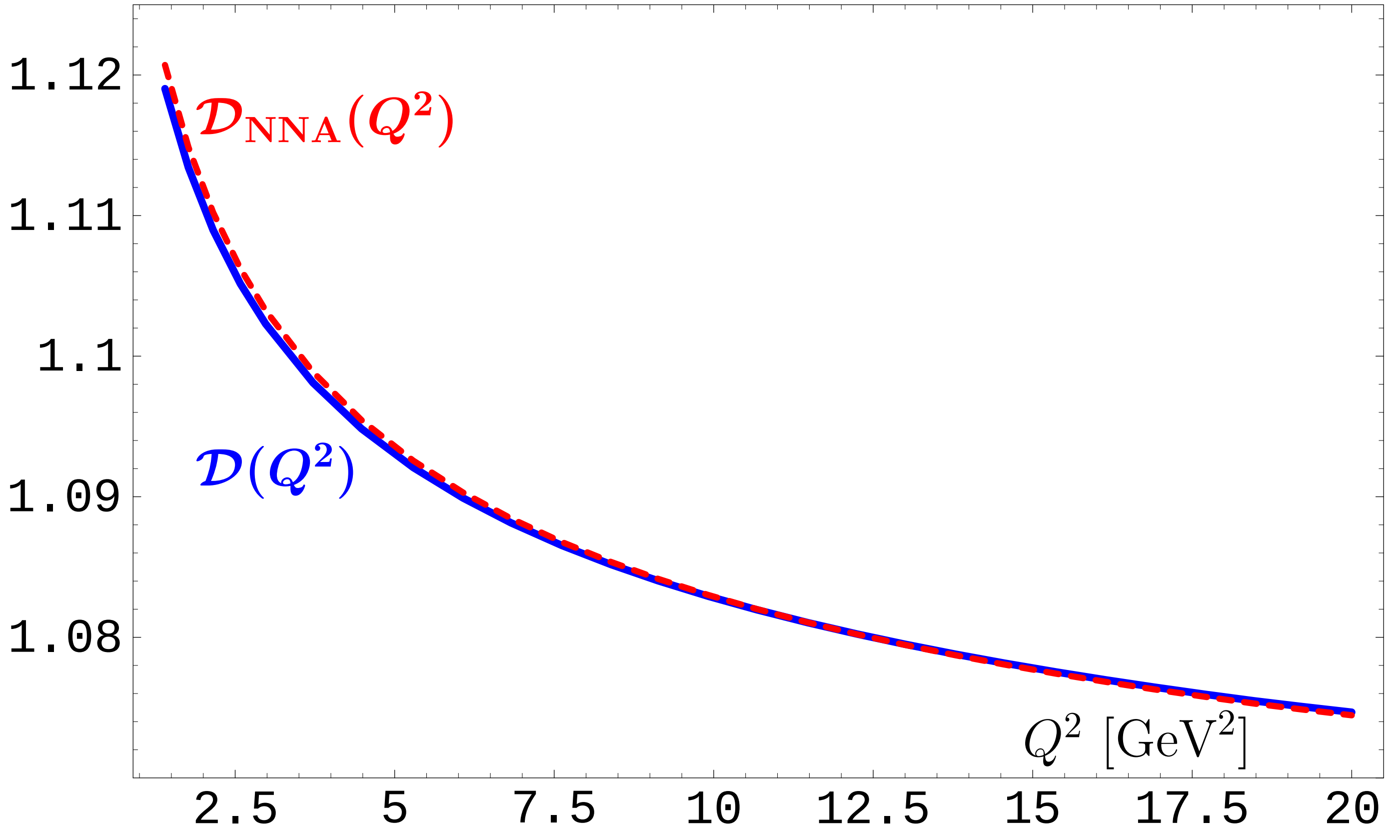
We apply it to resum APT series and obtain $\mathcal{D}_{\text{NNA}}(Q^2)$.

APT(E) for $\mathcal{D}(Q^2)$: Errors of modelling $P(t)$

Conclusion: **The result of resummation is stable to the variations of higher-order coefficients: deviation is of the order of 0.1%.**



Application

to

Higgs boson decay

Higgs boson decay into $b\bar{b}$-pair

This decay can be expressed in QCD by means of the correlator of quark scalar currents $J_{\text{S}}(x) = :\bar{b}(x)b(x):$

$$\Pi(Q^2) = (4\pi)^2 i \int dx e^{iqx} \langle 0 |\, T[\, J_{\text{S}}(x) J_{\text{S}}(0)\,]\, | 0 \rangle$$

in terms of discontinuity of its imaginary part

$$R_{\text{S}}(s) = \text{Im}\,\Pi(-s - i\epsilon)/(2\pi\, s)\,,$$

so that

$$\Gamma_{\text{H}\to b\bar{b}}(M_{\text{H}}) = \frac{G_F}{4\sqrt{2}\pi} M_{\text{H}}\, m_b^2(M_{\text{H}})\, R_{\text{S}}(s = M_{\text{H}}^2)\,.$$

FAPT(M) analysis of R_S

Running mass $m(Q^2)$ is described by the RG equation

$$m^2(Q^2) = \hat{m}^2 \alpha_s^{\nu_0}(Q^2) \left[1 + \frac{c_1 b_0 \alpha_s(Q^2)}{4\pi^2}\right]^{\nu_1} .$$

with RG-invariant mass $\hat{m}^2$ (for b-quark $\hat{m}_b \approx 8.53$ GeV) and $\nu_0 = 1.04$, $\nu_1 = 1.86$.

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with RG-invariant mass $\hat{m}^2$ (for b-quark $\hat{m}_b \approx 8.53$ GeV) and $\nu_0 = 1.04$, $\nu_1 = 1.86$. This gives us

$$\left[3\,\hat{m}_b^2\right]^{-1} \widetilde{D}_S(Q^2) = \alpha_s^{\nu_0}(Q^2) + \sum_{m>0} \frac{d_m}{\pi^m}\, \alpha_s^{m+\nu_0}(Q^2)\,.$$

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$$\left[3\,\hat{m}_b^2\right]^{-1} \widetilde{D}_{\mathbf{S}}(Q^2) = \alpha_s^{\nu_0}(Q^2) + \sum_{m>0} \frac{d_m}{\pi^m}\, \alpha_s^{m+\nu_0}(Q^2) \,.$$

In 1-loop FAPT(M) we obtain

$$\widetilde{\mathcal{R}}_{\mathbf{S}}^{(1);N}[L] = 3\hat{m}^2 \left[\mathfrak{A}_{\nu_0}^{(1);\text{glob}}[L] + \sum_{m>0}^{N} \frac{d_m}{\pi^m}\, \mathfrak{A}_{m+\nu_0}^{(1);\text{glob}}[L]\right]$$

FAPT(M) analysis of R_{S}

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$$\left[3\,\hat{m}_b^2\right]^{-1} \widetilde{D}_{\text{S}}(Q^2) = \alpha_s^{\nu_0}(Q^2) + \sum_{m>0} \frac{d_m}{\pi^m}\, \alpha_s^{m+\nu_0}(Q^2)\,.$$

In 2-loop FAPT(M) we obtain

$$\widetilde{\mathcal{R}}_{\text{S}}^{(2);N}[L] = 3\hat{m}^2 \left[\mathfrak{B}_{\nu_0,\nu_1}^{(2);\text{glob}}[L] + \sum_{m>0}^{N} \frac{d_m}{\pi^m}\, \mathfrak{B}_{m+\nu_0,\nu_1}^{(2);\text{glob}}[L]\right]$$

Model for perturbative coefficients

Coefficients of our series, $\tilde{d}_m = d_m/d_1$, with $d_1 = 17/3$:

Model	$\tilde{d}_1$	$\tilde{d}_2$	$\tilde{d}_3$	$\tilde{d}_4$	$\tilde{d}_5$
pQCD	1	7.42	62.3		—

Model for perturbative coefficients

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Model	$\tilde{d}_1$	$\tilde{d}_2$	$\tilde{d}_3$	$\tilde{d}_4$	$\tilde{d}_5$
pQCD	**1**	**7.42**	**62.3**		**—**
$c = 2.5,\ \beta = -0.48$	**1**	**7.42**	**62.3**		

We use model $\tilde{d}_n^{\text{mod}} = \dfrac{c^{n-1}\left(\beta\,\Gamma(n) + \Gamma(n+1)\right)}{\beta + 1}$

with parameters β and c estimated by known $\tilde{d}_n$ and with use of Lipatov asymptotics.

Model for perturbative coefficients

Coefficients of our series, $\tilde{d}_m = d_m/d_1$**, with** $d_1 = 17/3$**:**

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pQCD	**1**	**7.42**	**62.3**	**620**	**—**
$c = 2.5,\ \beta = -0.48$	**1**	**7.42**	**62.3**	662	**—**

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"PMS" model	—	—	64.8	547	7782

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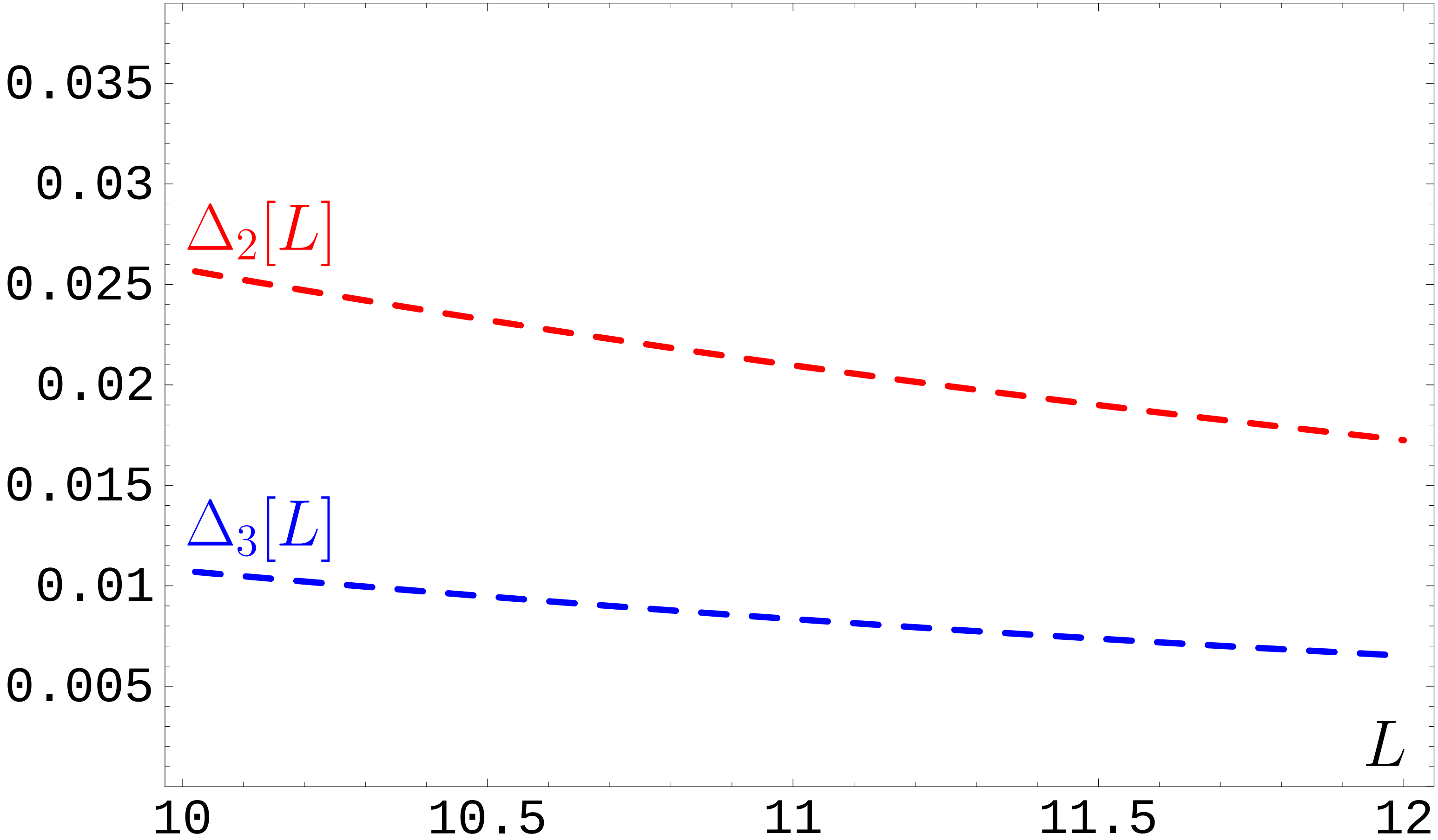
FAPT(M) for $\Gamma_{H\to\bar{b}b}(m_H)$: Truncation errors

We define relative errors of series truncation at Nth term:

$$\Delta_N[L] = 1 - \widetilde{\mathcal{R}}_{\mathsf{S}}^{(2;N)}[L]/\widetilde{\mathcal{R}}_{\mathsf{S}}^{(2;\infty)}[L]$$



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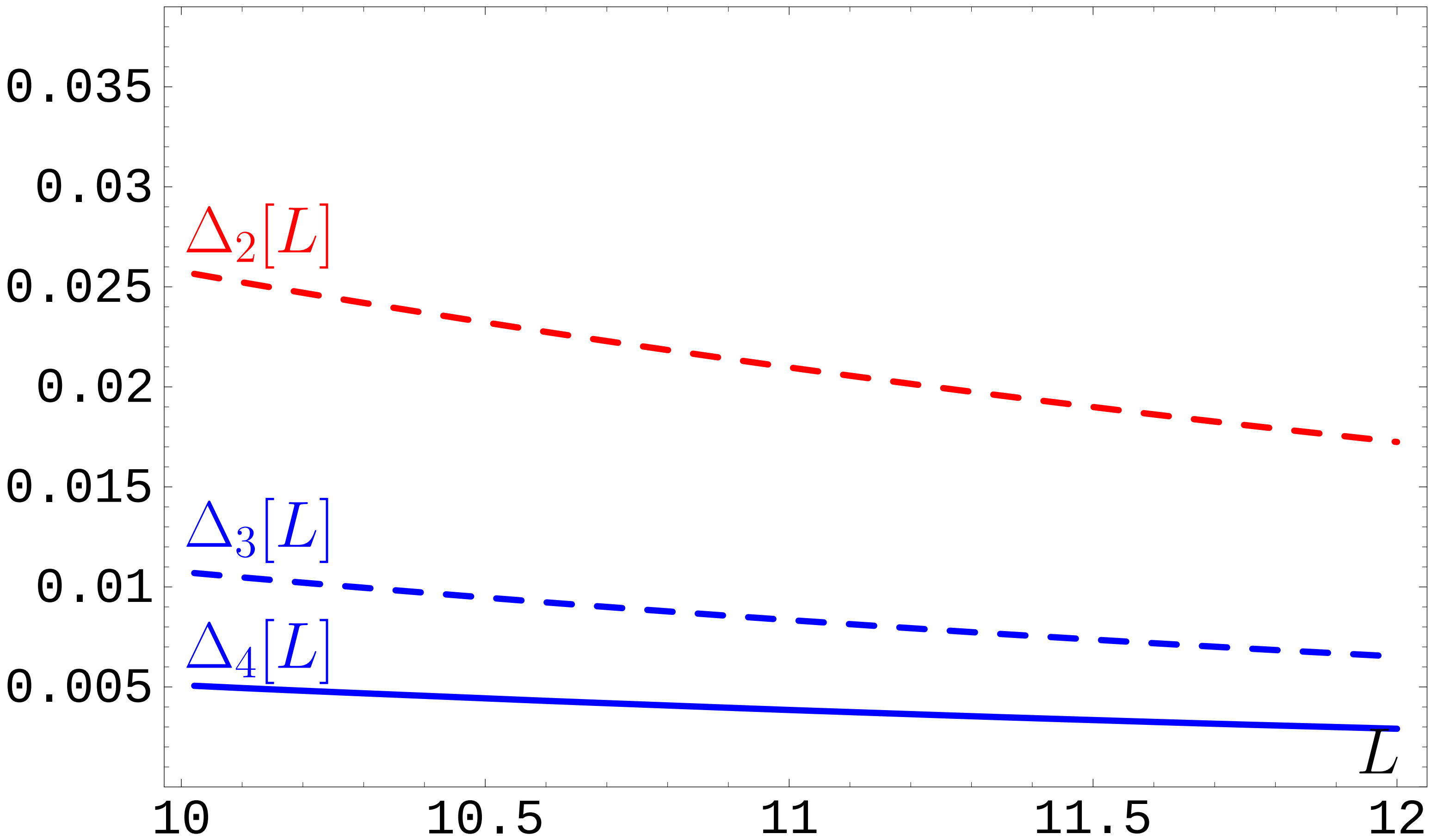

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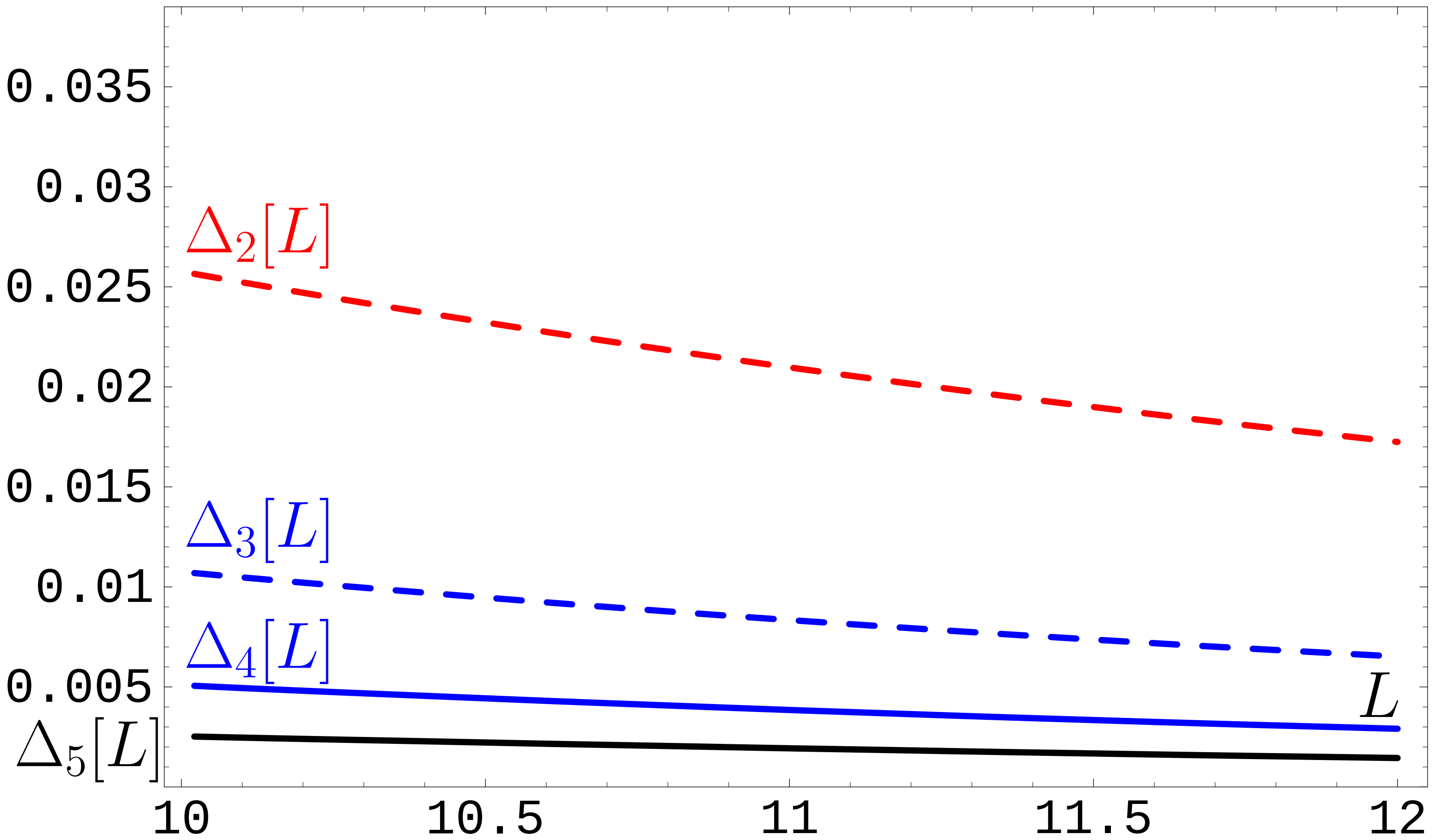

FAPT(M) for $\Gamma_{H\to\bar{b}b}(m_H)$*: Truncation errors*

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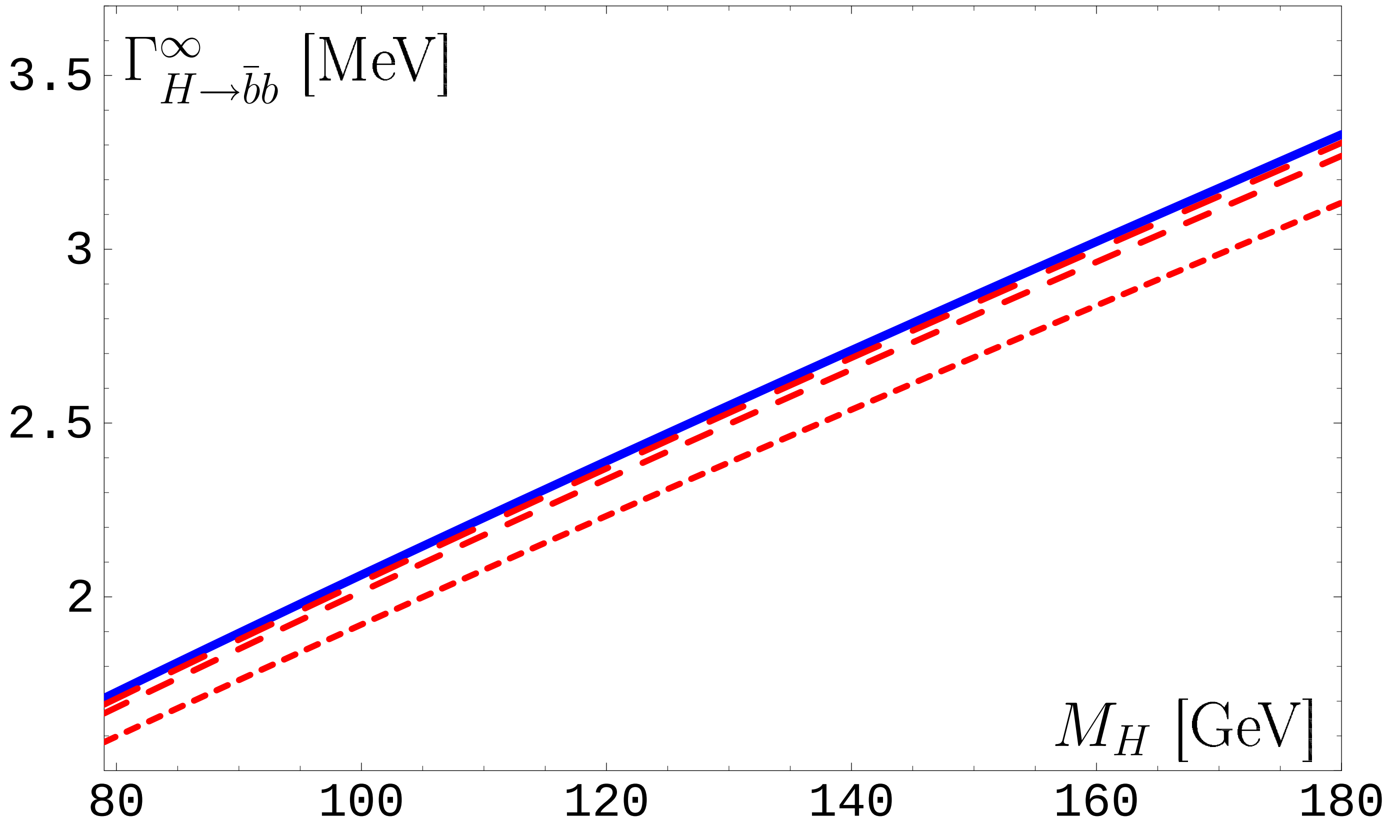
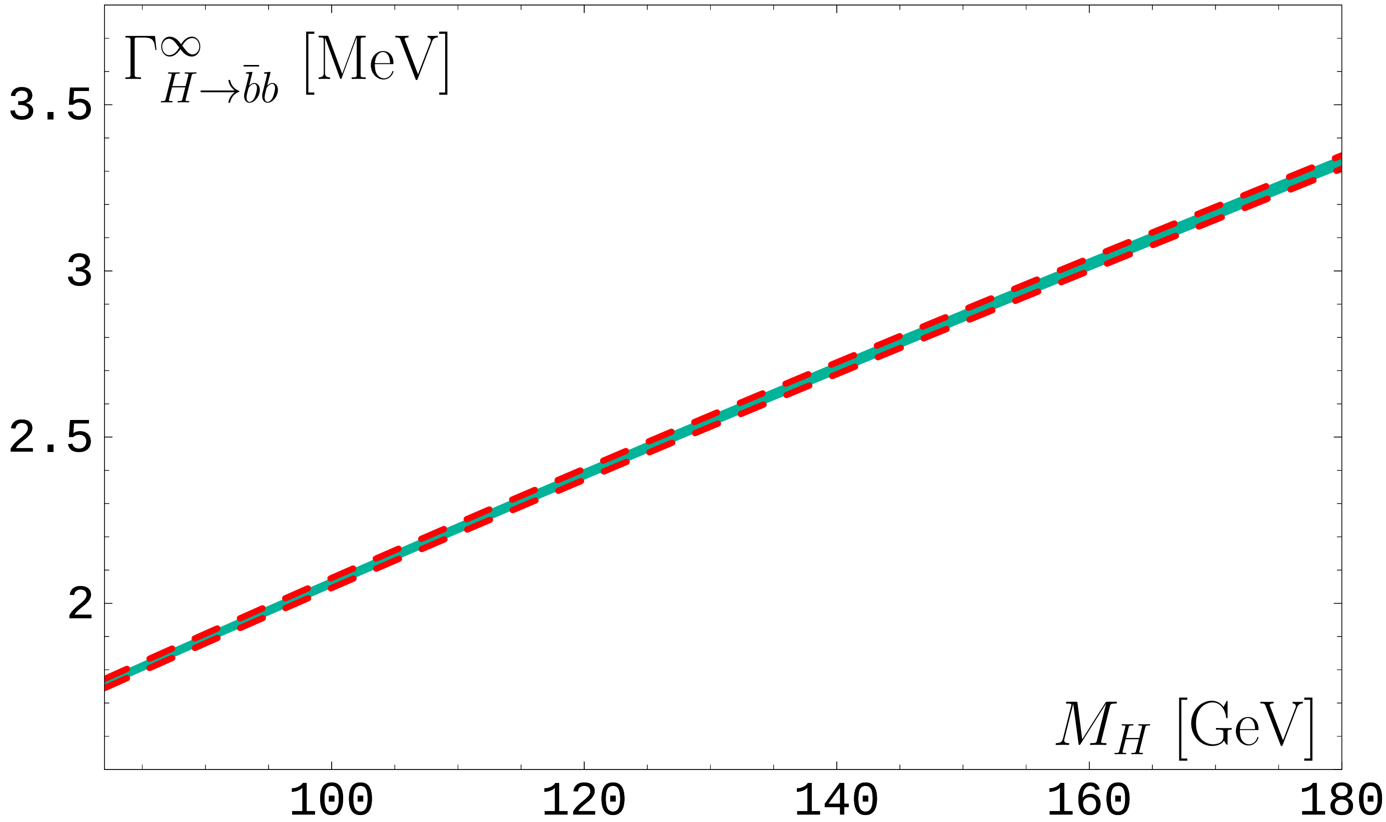
But profit will be tiny — instead of 0.5% one'll obtain 0.3%!



FAPT(M) for $\Gamma_{H\to\bar{b}b}(m_H)$: Truncation errors

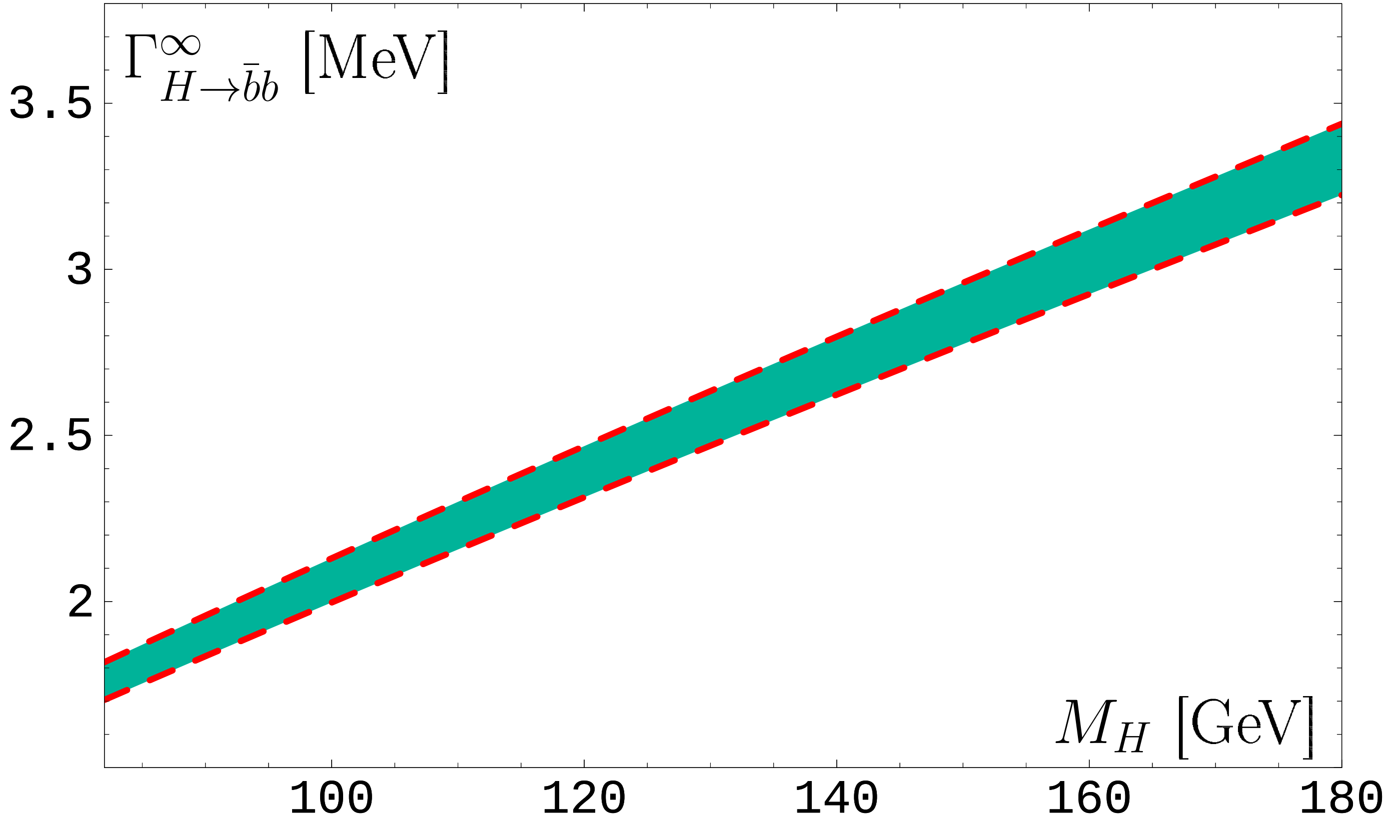
Conclusion: If we need accuracy of the order 0.5% — then we need to take into account up to the 4-th correction.

Note: uncertainty due to $P(t)$-modelling is small $\lesssim 0.6\%$.



FAPT(M) for $\Gamma_{H\to\bar{b}b}(m_H)$: Truncation errors

Conclusion: **If we need accuracy of the order 1% — then we need to take into account up to the 3-rd correction — in agreement with Kataev&Kim [0902.1442].**
Note: **RG-invariant mass uncertainty $\sim 2\%$.**



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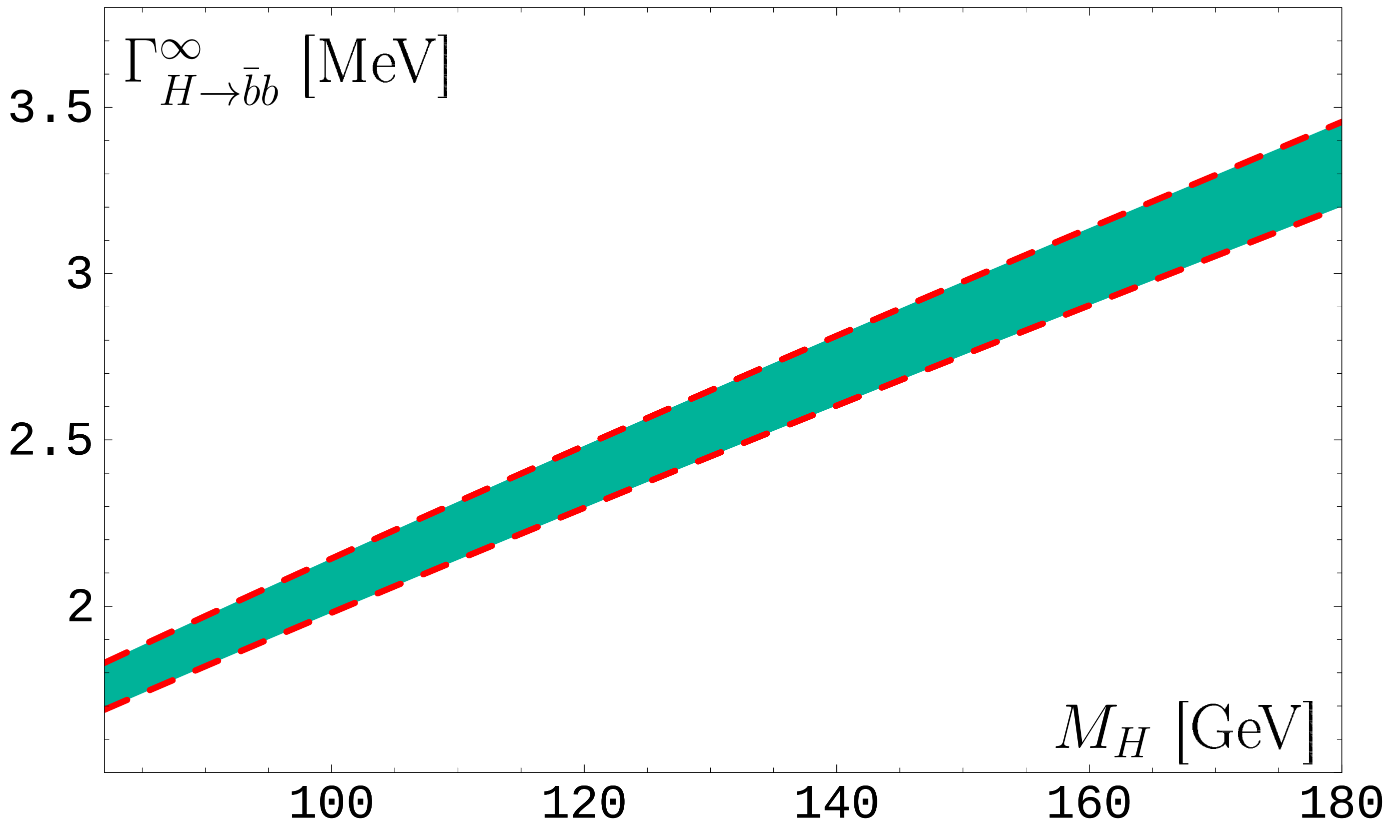
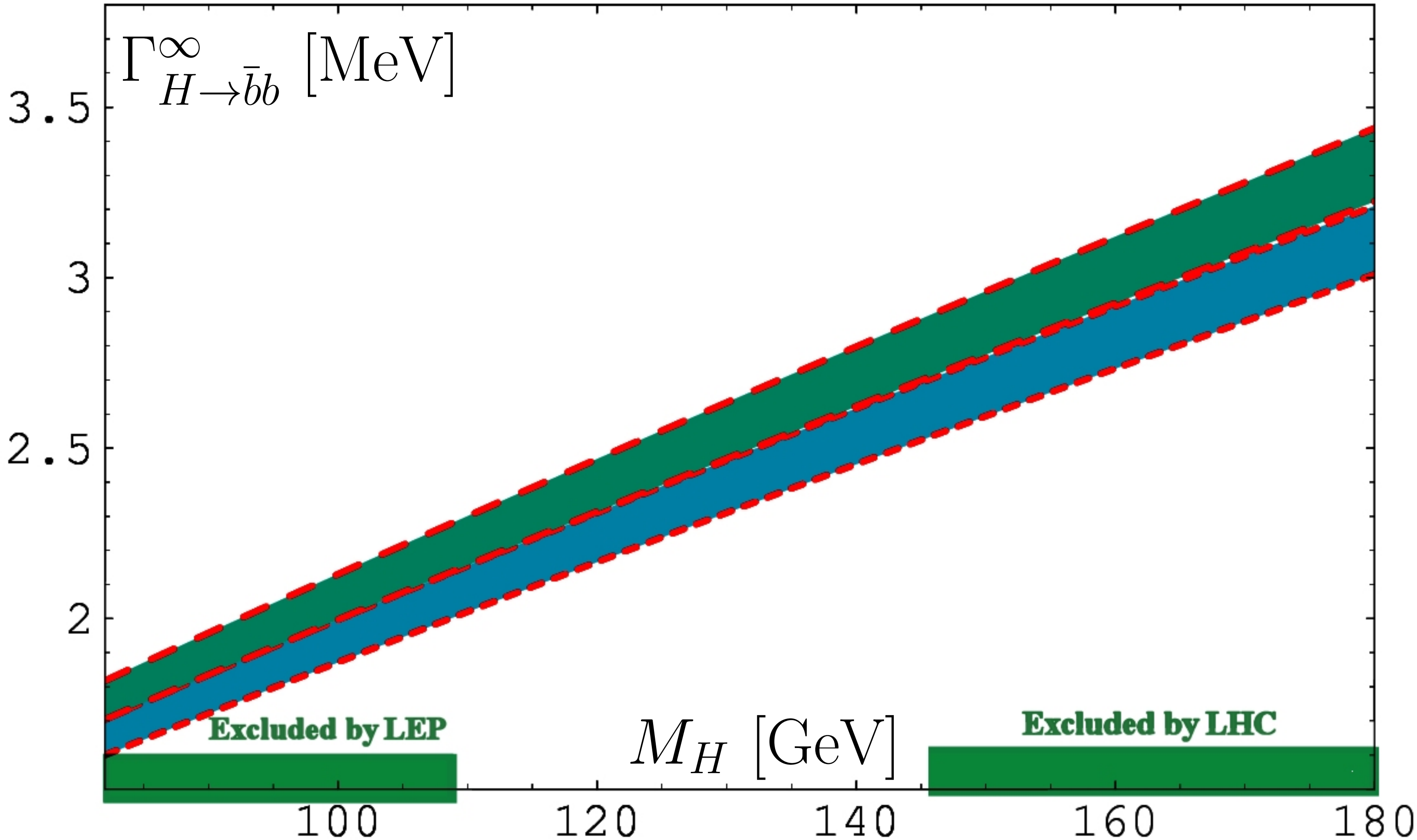
Note: **overall uncertainty $\sim 3\%$.**



Resummation for $\Gamma_{H\to\bar{b}b}(m_H)$: Loop orders

Comparison of 1- (upper strip) and 2- (lower strip) loop results.
We observe a 5% reduction of the two-loop estimate.



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